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Invariants of the General Linear Differential Equation and their Relation to the Theory of Continuous Groups.

By Charles L. Bouton.

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In the course of the last three decades Lie has created a general invariant theory for all continuous transformation groups which can be defined by differential equations. This far-reaching theory had many precursors, and among these Cayley's invariant theory has played a particularly important role in the mathematical development of our century. The deformation theory of surfaces, founded by Gauss and Minding, may also, as Lie has shown, be regarded as the invariant theory of a group, of an *infinite* group. On the other hand the theory of curvature of Euler and Monge is the invariant theory of the *finite* group of motions.

In the years 1870–72 Lie founded an invariant theory of the general group of all Contact Transformations, and at the same time, of certain infinite groups which he termed "Funktionengruppen." A development of these general investigations was his Theory of Transformation Groups (1874), and his General Invariant Theory of all Transformation Groups, the latter published in the years 1882–1884.

Laguerre and Brioschi in 1879 gave certain methods of finding invariants of linear differential equations; and Halphen, in his prize essay of 1882, gave a general method for finding such invariants, and applied it in the simpler cases. None of these mathematicians, however, refer to the work of an earlier writer on this subject, Cockle, who, as early as 1862, had found invariants of the general equation. The idea that all of these researches may be treated by the invariant theory of infinite groups belongs to Lie.

It is proposed to give in the following paper the derivation, according to Lie's method of continuous groups, of the invariants of the general linear differential equation of the n^{th} order.

The first mention of invariants of the linear differential equation seems to have been by Cockle in 1862. For the next fifteen or twenty years Cockle wrote at intervals on this subject. An account of his work will be found in Chapter 1. It may be thought that this review is rather more detailed than the importance of his results warrants, especially when compared with the amount of space given to the work of other writers. This has been done for two reasons. Cockle's investigations were given in a considerable number of comparatively short papers, and are therefore much more inaccessible than those of other writers on this subject, who gave their results in one or two memoirs. It therefore seemed desirable to bring the methods and results of this *first* writer together in one place. A second reason was that Cockle's examples, involving as they do both invariants and covariants for transformation of one variable at a time, seemed to be particularly well suited for illustrating the application of Lie's methods in the simpler cases, before taking up the general case. Cockle seems to have used only finite transformations, and to have actually found invariants for transformation of only *one* of the variables at a time. Then Laguerre* and Brioschi* in 1879 found certain invariants for transformation of both variables. Laguerre's method of finding "semi-invariants" for transformation of the dependent variable alone, is identical with the method given by Cockle nine years earlier. Laguerre made use of his invariants to remove the two terms of order next to the highest from the equation. Halphen in 1882† and 1883‡ carried the theory still further, and made most important applications of it. In 1884 Lie|| called attention to the fact that these invariants may all be found by his methods as the invariants of a certain infinite group. Forsyth§ in 1888 gave *all* the invariants and covariants of the equation in a *normal form*, together with a method of finding them for a form of the equation in which the term of order next to the highest is removed. For this latter form of the equation he gives the first five relative invariants. Forsyth's method is one which makes use of *infinitesimal transformations*, and the formulæ he deduces for the increments of the quantities involved are exactly the same as those which must be used in the method of continuous groups. Compare, for example, Forsyth's formulæ (12) and (13) (l. c., pp. 395, 397), with the formula (36) of this paper. In the latter part of his memoir Forsyth deduces partial differential equations of which his invariants are solutions. All of these equations (excepting those which have to do with the variables of the adjoined equation, which is not here considered) may be derived from the equations to be given later, but it has not been thought worth while to point out the identity in each case.

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* Comptes Rendus, t. 88 (1879), pp. 116–119; pp. 224–227.

In the following article, after reviewing Cockle's methods and results in Chapter 1, Lie's methods are used in Chapter 2 to find all of Cockle's results, and to verify some of the statements which he made without proof. Then in Chapters 3, 4, 5 the linear partial differential equations, of which the invariants are solutions, are deduced by Lie's methods. This is done for three cases, viz. for

1. The non-specialized equation.
2. The case in which the second term of the equation is removed.
3. Forsyth's normal form.

The transformations used in each case are the most general which leave the chosen form of equation unchanged. For the first two cases the general equa-

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* Bull. Soc. Math. de France, t. 7 (1879), pp. 105–108.

† Mémoires des Savants Etrangeres, t. 28, 2. series, 300 pp.

‡ Acta Math., t. 3 (1883), pp. 325–380.

|| Math. Anal., Bd. 24 (1884), p. 573.

§ Phil. Trans., 1888, I, pp. 377–489.

tions have been solved for a few special cases of the constants involved. For case 3 the complete solution in explicit form is given, it being then comparatively easy to integrate the equations.

It is believed that the linear differential equations of cases 1 and 2, namely, the equations numbered (30) and (40), are here given for the first time.

CHAPTER 1.

Cockle's Work.

§1. The earliest notice which I have been able to find of the fact that there are functions of the coefficients $a_1, \ldots, a_n$ of the homogeneous linear differential equation

$$\frac{d^ny}{dx^n} + na_1\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{2}a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = 0, \tag{1}$$

(where $a_1, \ldots, a_n$ are functions of x alone), which remain invariant for changes of the variables which do not change the form or order of the equation, is in an article by Sir James Cockle on "The Correlations of Analysis," dated November, 1862.* Cockle there shows that when, in the general equation (1), y is subjected to the transformation, $y = v\,.\,u(x)$, where v is the new dependent variable, that

$$a_1^2 - a_2 + a_1' \quad \text{and} \quad 2a_1^3 - 3a_1a_2 + a_3 - a_1''$$

are absolute invariants. The method of proof was to express the new coefficients $A_1, \ldots, A_n$ in terms of $a_1, \ldots, a_n$, u, and the derivatives of u, and then to eliminate u and its derivatives between the resulting equations. In 1870† Cockle calls such invariant functions as the above *criticoids*, these two being respectively the *quadricriticoid* and *cubicriticoid.* The method of elimination may be used in obtaining criticoids of higher degree, but this method has its disadvantages, namely, "want of directness and generality, and a failure of proof of the existence of criticoids other than those actually obtained or deducible from them by combination" (l. c., p. 202). Then, on pp. 209–210 of this same memoir, Cockle gives the method which he regards as the most satisfactory and final, for he seems to have given no other methods. This method "gives by a direct process results perfectly general, and makes manifest the existence of criticoids of all degrees." The process consists in eliminating the *second term* of the differential equation (1) by the substitution $y = ve^{-\int a_1 dx}$. The coefficients of the transformed equation divided by their respective binomial coefficients, viz. $B_2, B_3, \ldots, B_n$, are then $n-1$ primary criticoids of degree 2 to n respectively. The *quarticriticoid* is

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* **Phil. Mag., IV Series, vol. 24, p. 532, §2.**

† **Phil. Mag., vol. 39, p. 202.**

$$a_4 - 4a_1 a_3 - 3a_2^2 + 12a_1^2 a_2 - 6a_1^4 - a_1'''.$$

The following is then Cockle's final method for finding criticoids, carried as far as finding the first three: In (1) substitute $y = ve^{-\int a_1 dx}$. Then we have

$$\frac{d^m y}{dx^m} = \overset{m}{\underset{0}{\sum}}{}_s \frac{m!}{(m-s)!\, s!} \frac{d^s \left(e^{-\int a_1 dx}\right)}{dx^s} \frac{d^{m-s} v}{dx^{m-s}},$$

and we find that the transformed equation becomes

$$\frac{d^n v}{dx^n} + \frac{n(n-1)}{2!} B_2 \frac{d^{n-2} v}{dx^{n-2}} + \frac{n(n-1)(n-2)}{3!} B_3 \frac{d^{n-3} v}{dx^{n-3}} + \ldots + B_n v = 0, \qquad (1')$$

where

$$B_2 = a_2 - a_1^2 - a_1', \quad B_3 = a_3 - 3a_2 a_1 + 2a_1^3 - a_1'',$$

$$B_4 = a_4 - 4a_3 a_1 + 6a_2 a_1^2 - 6a_2 a_1' - 3a_1^4 + 6a_1^2 a_1' + 3a_1'^2 - a_1''',$$

or

$$B_4 = a_4 - 4a_3 a_1 + 12a_2 a_1^2 - 3a_2^2 - 6a_1^4 - a_1''' + 3B_2^2.$$

These functions B_k of the coefficients of the equation (1) are then invariant for any transformation of the dependent variable y, of the form $y = v \cdot u(x)$, where u is an arbitrary function of x. Cockle calls these functions B_k the *primary criticoids.**

It is easy to show that we have in general (although Cockle nowhere explicitly mentions this),

$$B_k = e^{\int a_1 dx} \overset{k}{\underset{0}{\sum}}{}_s \frac{k!}{(k-s)!\, s!} a_{k-s} \frac{d^s \left(e^{-\int a_1 dx}\right)}{dx^s}.$$

In a paper on "Hyperdistributives" † Cockle proves that, if we write

$$\theta_m(a) = \frac{d^m \log a}{dx^m},$$

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* Phil. Mag., vol. 39 (1870), p. 210, "On Criticoids."

† Phil. Mag., vol. 43 (1872), p. 300.

and if after the differentiation we write

$$\frac{1}{a}\,\frac{d^r a}{dx^r} = a_r\,,$$

and call the resulting function $\theta_m\,(a_1,\,a_2,\,\ldots\,,\,a_m)$, then

$$\theta_m\,(a_1,\,\ldots\,,\,a_m) - \frac{d^{m-1}a_1}{dx^{m-1}}$$

is the entire criticoidal function of the m^{th} order.

In another paper* it is stated that the derivative of a criticoid with respect to the independent variable x is a criticoid.

On page 203 of the article in volume 39, Cockle states that he did not attempt to employ an *operative symbol.* By "operative symbol" he means a differential operator which, when applied to the criticoid, reduces it to zero. The operative symbol is of the nature of a partial differential equation, which the function it reduces to zero satisfies. An example of such a symbol, taken from one of Cockle's earlier papers, will be given later.

§2. After having found in 1862 the two differential invariants mentioned above, Cockle, in 1864,† gave a *differential covariant* for the equation of the third order,

$$\frac{d^3y}{dx^3} + 3a_1\frac{d^2y}{dx^2} + 3a_2\frac{dy}{dx} + a_3 y = 0\,.$$

It is shown by direct computation that

$$B_2\frac{d^2y}{dx^2} + [2B_2a_1 - B_3 - B_2']\frac{dy}{dx} + [B_2a_2 - (B_3 + B_2')\,a_1]\,y\,,$$

where B_2 and B_3 have the values already given, is a *relative differential covariant* for the substitution $y = v\,.\,u(x)$. The new function is equal to the old multiplied by u. Cockle calls this covariant the "differential Hessian," from its resemblance to a Hessian of an algebraic form. He seems to have found it by trial, for he says (p. 228): "I first deduced the differential Hessian by seeking an expression for which the coefficients of transformation and of substitution should be the same, and I afterward found that an operator, to wit Δ, reduced it to zero." This operative symbol Δ is

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* Phil. Mag., vol. 27 (1864), p. 227.

† Phil. Mag., vol. 27, pp. 225–228.

$$\Delta = \delta - \frac{d}{da_1} - 2a_1\frac{d}{da_2} - 3a_2\frac{d}{da_3},$$

or as it is given in a later paper,*

$$\Delta = \delta - \left[a_0\frac{d}{da_1}\right] - 2\left[a_1\frac{d}{da_2}\right] - 3\left[a_2\frac{d}{da_3}\right],$$

where a_0 is a coefficient multiplied into $\frac{d^3y}{dx^3}$ to make the given differential equation homogeneous in a. Here δ is defined by means of the equation

$$\delta\frac{d^my}{dx^m} = m\frac{d^{m-1}y}{dx^{m-1}}.$$

The symbols in the brackets are supposed to be commutative with $\frac{d}{dx}$, so that we have, for example,

$$\begin{aligned}\left[a_1\frac{d}{da_2}\right](a_1a_2') &= a_1\left[a_1\frac{d}{da_2}\right]a_2' = a_1\left[a_1\frac{d}{da_2}\right]\frac{da_2}{dx} = a_1\frac{d}{dx}\left[a_1\frac{d}{da_2}\right]a_2\\ &= a_1\frac{da_1}{dx} = a_1a_1'.\end{aligned}$$

It will be useful, in view of comparisons to be made later, to express this symbol Δ in terms of partial differential coefficients. Since

$$\left[a_{i-1}\frac{d}{da_i}\right]a_i^{(k)} = \frac{d^k}{dx^k}\left[a_{i-1}\frac{d}{da_i}\right]a_i = a_{i-1}^{(k)},$$

and

$$\left[a_{i-1}\frac{d}{da_i}\right](a_i^{(k)})^m = m\,(a_i^{(k)})^{m-1}\left[a_{i-1}\frac{d}{da_i}\right]a_i^{(k)} = m\,(a_i^{(k)})^{m-1}a_{i-1}^{(k)},$$

we have

$$\left[a_{i-1}\frac{d}{da_i}\right] = \sum_0^k a_{i-1}^{(k)}\frac{\partial}{\partial a_i^{(k)}}, \qquad (i = 1,\ 2,\ 3)$$

and

$$\delta = \sum_1^m my^{(m-1)}\frac{\partial}{\partial y^{(m)}}.$$

Hence the symbol Δ is equivalent to

$$\sum_1^m my^{(m-1)}\frac{\partial}{\partial y^{(m)}} - \sum_0^k a_0^{(k)}\frac{\partial}{\partial a_1^{(k)}} - 2\sum_0^k a_1^{(k)}\frac{\partial}{\partial a_2^{(k)}} - 3\sum_0^k a_2^{(k)}\frac{\partial}{\partial a_3^{(k)}}.$$

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* Phil. Mag., vol. 28 (1864), pp. 205-206.

The upper limits in the different summations are to be chosen large enough so that a farther increase would yield only zero when the operator is applied to the function in question. For the covariant under consideration these developments need only be carried as far as follows:

$$y\frac{\partial}{\partial y'} + 2y'\frac{\partial}{\partial y''} - \frac{\partial}{\partial a_1} - 2a_1\frac{\partial}{\partial a_2} - 2a_1'\frac{\partial}{\partial a_2'} - 3a_2\frac{\partial}{\partial a_3}\,.$$

We readily verify that this operator reduces the covariant to zero.

The foregoing differential covariant is not the simplest, as Cockle showed in a paper of August, 1865.* Using the notation

$$a_0\frac{d^m y}{dx^m} + ma_1\frac{d^{m-1}y}{dx^{m-1}} + \frac{m(m-1)}{2}a_2\frac{d^{m-2}y}{dx^{m-2}} + \cdots\cdot + a_m y \equiv y_m,$$

and designating the differential covariants as *covaroids*, he there states, without proof, that "the functions y_n, y_m, and $\frac{d^p y_q}{dx^p}$ are covaroids for all values of m, n, p, and q, and consequently any functions of B_2, B_3, and the higher criticoids, and also of any number of values of y_m and of $\frac{d^p y_q}{dx^p}$ are covaroids, and *vice versa*. Accordingly the covaroid which I have called the differential Hessian may, after a slight modification, be put under the form $B_2 y_2 - (B_3 + B_2')\,y_1$, and is only one of an infinite number of quadricovaroids." This is a very important statement, especially the *vice versa,* as that indicates that Cockle had found, or thought he had found, the complete solution of the problem of finding *all* the covariants (and hence all the invariants) of a homogeneous linear differential equation of the n^{th} order, for a transformation of the dependent variable of the form $y = v\,.\,u(x)$. This was in 1865. As no proof of this statement is given, we must here regard it as an induction, and examine its truth later.

§3. In 1875† Cockle gave a relative invariant of the general equation (1) for change of the independent variable. The first four pages of the article are devoted to finding the formulæ for change of the independent variable, and the method used, although ingenious, appears very artificial and cumbrous. Having this formula, the work of finding the invariant, or *differential criticoid,* is comparatively short. The result is that if in (1) the independent variable be changed from x to t, where $t = \phi(x)$, so that (1) becomes

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*Phil. Mag., vol. 30, pp. 347–348.

† Phil. Mag., vol. 50, p. 440.

$$\frac{d^n y}{dt^n} + nA_1 \frac{d^{n-1}y}{dt^{n-1}} + \frac{n(n-1)}{2!} A_2 \frac{d^{n-2}y}{dt^{n-2}} + \ldots . + A_n y = 0,$$

where $A_1, \ldots\ldots, A_n$ are functions of t alone, that then

$$\left[\frac{dA_1}{dt} + \frac{3n-1}{2(n-1)} A_1^2 - \tfrac{3}{2} \cdot \frac{n-1}{n-2} A_2\right] dt^2 = \left[\frac{da_1}{dx} + \frac{3n-1}{2(n-1)} a_1^2 - \tfrac{3}{2} \cdot \frac{n-1}{n-2} a_2\right] dx^2,$$
$$n > 2,$$

"and either side is a differential criticoid which we may term a differential quadricriticoid." Since y remains unchanged, this amounts to saying that

$$\left[a_1' + \frac{3n-1}{2(n-1)} a_1^2 - \frac{3(n-1)}{2(n-2)} a_2\right] \frac{1}{y'^2}, \qquad n > 2$$

is an absolute covariant for the transformation $t = \phi(x)$. "The differential varies from the ordinary criticoid in this, that the corresponding coefficients of the former contain n, the order of the differential equation, while those of the latter are free from n."

I cannot find that Cockle has published any other invariant for transformation of the independent variable, nor any for the simultaneous transformation of both dependent and independent variables. Harley* mentions, however, that Cockle had in a letter suggested the possibility of the existence of such invariants.

§4. *Resumé of Cockle's Results.*

In 1862 Cockle gave in explicit form two invariants of the general equation (1) for transformation of the dependent variable alone. In 1870 he gave a method of finding a system of invariants, for change of the dependent variable, from which all others may be derived by differentiation, although he does not prove this system to be *complete*, merely mentioning that other invariants may be obtained by differentiating those already found. In 1864 he gave a covariant for the equation of the third order, and in 1865 he stated how all the covariants of the general equation may be found for transformation of the dependent variable alone. In 1875 Cockle gave a single invariant of the general equation for transformation of the independent variable alone. In all of his articles he uses only *finite* transformations, and in the main his methods seem to have been elimination methods, although it is not always clear how he first obtained his results.

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* Royal Soc. Proc., vol. 38 (1884), p. 57.

CHAPTER 2.

Cockle's Results obtained by Lie's Methods.

§5. It will now be shown how Cockle's results would be obtained by Lie's method of continuous groups. The outline of the method as applied to this particular problem was given by Lie in 1884.* The following derivation follows that outline, and is made rather more general than is necessary to obtain the few invariants actually found by Cockle, so that the truth of some of his statements may be examined.

Take first the case in which the dependent variable alone is transformed. The variables x and y are to be transformed by means of

$$x_1 = x, \quad y_1 = y\psi(x),$$

where x_1, y_1 are the transformed variables, and ψ is an arbitrary analytic function of x alone. All of these transformations evidently form an infinite continuous group defined by differential equations.

We get the infinitesimal transformation by writing $\psi(x) = 1 + \phi(x) \, . \, \delta t$; then $\delta x = 0$. $\delta y = y\phi(x) \, . \, \delta t$, and the symbol of the infinitesimal transformation is

$$\phi(x)\, y \frac{\partial f}{\partial y} \, .$$

We are to subject the equation

$$y^{(n)} + na_1 y^{(n-1)} + \frac{n(n-1)}{2!} a_2 y^{(n-2)} + \dots\dots + a_n y = 0 \tag{2}$$

to this transformation, and find functions of the coefficients $a_1, \dots\dots, a_n$ and their derivatives which are invariant. To this end we must compute the increments of $a_1, \dots\dots, a_n$, and of the various derivatives, and form the *extended*

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* Math. Annal., Band 24, p. 573.

transformation. We have

$$\delta y^{(k)} = \delta \frac{dy^{(k-1)}}{dx} = \frac{d}{dx}\,\delta y^{(k-1)} - y^{(k)} \frac{d\delta x}{dx} = \frac{d}{dx}\,\delta y^{(k-1)}$$

$$= \frac{d^2}{dx^2}\,\delta y^{(k-2)} = \ldots\ldots = \frac{d^k}{dx^k}(\delta y),$$

$$\delta y^{(k)} = \frac{d^k}{dx^k}\left[\phi(x)\, y\right]\delta t = \sum_{m=0}^{k} \frac{k!}{(k-m)!\, m!}\,\phi^{(m)} y^{(k-m)}\,\delta t, \qquad (3)$$

by Euler's theorem for the differentiation of a product. Let us take the variation of (2):

$$\delta y^{(n)} + na_1\,\delta y^{(n-1)} + \frac{n(n-1)}{2!}\,a_2\delta y^{(n-2)} + \ldots\ldots + a_n\delta y$$

$$+ ny^{(n-1)}\delta a_1 + \frac{n(n-1)}{2}\,y^{(n-2)}\delta a_2 + \ldots\ldots + ny'\,\delta a_{n-1} + y\delta a_n = 0.$$

On substituting the values of $\delta y^{(n-k)}$ as given by (3), this becomes

$$\phi\delta t \sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!}\, a_k y^{(n-k)} + \sum_{k=1}^{n} \frac{n!}{(n-k)!\, k!}\, y^{(n-k)}\delta a_k$$

$$+ \sum_{k=0}^{n-1} \frac{n!}{(n-k)!\, k!}\, a_k \sum_{m=1}^{n-k} \frac{(n-k)!}{(n-k-m)!\, m!}\, \phi^{(m)} y^{(n-k-m)}\,\delta t = 0.$$

Here and in what follows we must understand $a_0 = 1$. Making use of (2) the first sum vanishes. Then changing the letters and order of summation in the last double sum, the preceding equation becomes

$$\sum_{k=1}^{n} \frac{n!}{(n-k)!\, k!}\, y^{(n-k)} \left[\delta a_k + \sum_{m=0}^{k-1} \frac{k!}{(k-m)!\, m!}\, \phi^{(k-m)} a_m\,\delta t\right] = 0.$$

This equation must subsist identically for all values of $y^{(n-k)}$, and hence we must have

$$\delta a_k = -\sum_{m=0}^{k-1} \frac{k!}{(k-m)!\, m!}\, \phi^{(k-m)} a_m\,\delta t = -\sum_{i=1}^{k} \frac{k!}{(k-i)!\, i!}\, a_{k-i}\,\phi^{(i)}\,\delta t. \qquad (4)$$

$$(k = 1, 2, \ldots\ldots, n)$$

We require not only the increments of $a_1, \ldots\ldots, a_n$, but also the increment of $a_k^{(s)} = \dfrac{d^s a_k}{dx^s}$, for $k = 1, 2, \ldots\ldots, n$, and all values of s. We have, as in deducing (3),

$$\delta a_k^{(s)} = \delta \left(\frac{d a_k^{(s-1)}}{dx} \right) = \frac{d}{dx} (\delta a_k^{(s-1)}) = \ldots\ldots = \frac{d^s}{dx^s} (d a_k),$$

whence

$$\delta a_k^{(s)} = -\sum_{m=0}^{k-1} \frac{k!}{(k-m)!\ m!} \frac{d^s}{dx^s} (\phi^{(k-m)} a_m)\, \delta t. \tag{5}$$

We have further

$$\frac{d^s}{dx^s} (\phi^{(k-m)} a_m) = \sum_{j=0}^{s} \frac{s!}{(s-j)!\, j!} \phi^{(k-m+j)} a_m^{(s-j)} = \sum_{i=k-m}^{s+k-m} \frac{s!\ a^{(s+k-m-i} \phi^{(i)}}{(s+k-m-i)!\ (i-k+m)!},$$

and consequently,

$$\left.\begin{aligned} \delta a_k^{(s)} &= -\sum_{m=0}^{k-1} \sum_{i=k-m}^{s+k-m} \frac{k!\ s!\ a_m^{(s+k-m-i)}\, \phi^{(i)}\, \delta t}{(k-m)!\ m!\ (s+k-m-i)!\ (i-k+m)!}, \\ \delta a_k^{(s)} &= -\sum_{m=1}^{k} \sum_{i=m}^{s+m} \frac{k!\ s!\ a_{k-m}^{(s+m-i)}\, \phi^{(i)}\, \delta t}{(k-m)!\ m!\ (s+m-i)!\ (i-m)!}. \end{aligned}\right\} \tag{6}$$

$$(k = 1, 2, \ldots\ldots, n;\ s = 0, 1, 2, \ldots\ldots)$$

These formulæ include (4) as a special case.

§6. Before considering the general case, let us investigate whether there are any invariants involving only a_1 and a_2 and their derivatives, and if so, how many. (6) gives:

$$\delta a_1 = -\phi'\, \delta t, \quad \delta a_1^{(s)} = -\phi^{(s+1)}\, \delta t, \quad \delta a_2 = -(2 a_1 \phi' + \phi'')\, \delta t,$$

$$\delta a_2^{(s-1)} = -\left[2 \sum_{i=1}^{s} \frac{(s-1)!\ a_1^{(s-i)}}{(s-i)!\ (i-1)!} \phi^{(i)} + \phi^{(s+1)} \right] \delta t.$$

Hence the extended transformation is

$$\phi y \frac{\partial f}{\partial y} - \sum_{s=0}^{\sigma} \phi^{(s+1)} \frac{\partial f}{\partial a_1^{(s)}} - \sum_{s=1}^{\sigma} \left\{ 2 \sum_{i=1}^{s} \frac{(s-1)!}{(s-i)!\ (i-1)!} a_1^{(s-i)} \phi^{(i)} + \phi^{(s+1)} \right\} \frac{\partial f}{\partial a_2^{(s-1)}}.$$

Any invariant must be a solution of the partial differential equation obtained by equating this transformation to zero, and that whatever the function ϕ may be. This condition is therefore equivalent to the following set of equations obtained by equating the coefficients of the separate derivatives of ϕ to zero:

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial y}\,, \\
0 &= \frac{\partial f}{\partial a_1} + 2a_1 \frac{\partial f}{\partial a_2} + 2\sum_{2}^{\sigma}{}_{s}\, a_1^{(s-1)} \frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1'} + \frac{\partial f}{\partial a_2} + 2\sum_{2}^{\sigma}{}_{s}\, (s-1)\, a_1^{(s-2)} \frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + 2\sum_{3}^{\sigma}{}_{s}\, \frac{(s-1)!}{(s-3)!\;2!}\, a_1^{(s-3)} \frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1'''} + \frac{\partial f}{\partial a_2''} + 2\sum_{4}^{\sigma}{}_{s}\, \frac{(s-1)!}{(s-4)!\;3!}\, a_1^{(s-4)} \frac{\partial f}{\partial a_2^{(s-1)}}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial f}{\partial a_1^{(\sigma-1)}} + \frac{\partial f}{\partial a_2^{(\sigma-2)}} + 2a_1 \frac{\partial f}{\partial a_2^{(\sigma-1)}}\,, \\
0 &= \frac{\partial f}{\partial a_1^{(\sigma)}} + \frac{\partial f}{\partial a_2^{(\sigma-1)}}\,.
\end{aligned}
$$

Now, according to Lie's theory,* this system of equations must be a *complete* system, of which the solutions are the invariants sought. It is also easy to verify directly by combining these equations that they form a complete system. Neglecting the first equation, which shows only that the invariant cannot involve y, the number of the variables is $\sigma + 1 + \sigma = 2\sigma + 1$. The number of equations is $\sigma + 1$, and hence there are exactly $(2\sigma + 1) - (\sigma + 1) = \sigma$ independent solutions, since the equations are obviously linearly independent. One solution is found to be

$$I_2 = a_2 - a_1^2 - a_1'.$$

This is Cockle's quadricriticoid. Since x is not transformed, $\frac{dI_2}{dx}$ must be invariant, for if u and v are any two invariants of the linear differential equation, then $\frac{dv}{du}$ is also invariant.† Hence the σ solutions of the foregoing system may be chosen as

$$I_2,\ \frac{dI_2}{dx},\ \frac{d^2I_2}{dx^2},\ \ldots\ldots,\ \frac{d^{\sigma-1}I_2}{dx^{\sigma-1}};$$

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* Math. Annal., Band 24.

† As a proof of this, for the general case is given later, a special proof is not given here. See §16.

for these invariants are linearly independent of one another, and involve only the variables of the system. We may say then that there is only one *essential* invariant involving only a_1 and a_2 and their derivatives to any order, as all others may be derived from this one by successive differentiation with regard to x. (Compare Cockle's statement that "the differentials of critical functions with regard to the independent variable are critical.")

§7. Next consider the invariants involving only a_1, a_2, a_3 and their derivatives. We have, in addition to the values already given,

$$\delta a_3 = -(3a_2\phi' + 3a_1\phi'' + \phi''')\,\delta t,$$

$$\delta a_3^{(s-2)} = -\Big[3\sum_0^{s-2}{}_i \frac{(s-2)!}{(s-2-i)!\;i!}\,a_2^{(s-2-i)}\phi^{(i+1)} + 3\sum_0^{s-2}{}_i \frac{(s-2)!}{(s-2-i)!\;i!}\,a_1^{(s-2-i)}\phi^{(i+2)} + \phi^{(s+1)}\Big]\,\delta t.$$

Forming now the extended transformation as before, and writing the first three of the resulting equations, we have

$$0 = \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + 2\sum_2^{\sigma}{}_s\, a_1^{(s-1)}\frac{\partial f}{\partial a_2^{(s-1)}} + 3\sum_3^{\sigma}{}_s\, a_2^{(s-2)}\frac{\partial f}{\partial a_3^{(s-2)}},$$

$$0 = \frac{\partial f}{\partial a_1'} + \frac{\partial f}{\partial a_2} + 3a_1\frac{\partial f}{\partial a_3} + 2\sum_2^{\sigma}{}_s\,(s-1)\,a_1^{(s-2)}\frac{\partial f}{\partial a_2^{(s-1)}} + 3\sum_3^{\sigma}{}_s\,\left[(s-2)\,a_2^{(s-3)} + a_1^{(s-2)}\right]\frac{\partial f}{\partial a_3^{(s-2)}},$$

$$0 = \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + \frac{\partial f}{\partial a_3} + 2\sum_3^{\sigma}{}_s\,\frac{(s-1)(s-2)}{2}\,a_1^{(s-3)}\frac{\partial f}{\partial a_2^{(s-1)}} + 3\sum_3^{\sigma}{}_s\left[\frac{(s-2)(s-3)}{2}\,a_2^{(s-4)} + (s-2)\,a_1^{(s-3)}\right]\frac{\partial f}{\partial a_3^{(s-2)}}.$$

The remaining $\sigma - 2$ equations involve differential coefficients of f with regard to $a_1''', \ldots, a_1^{(\sigma)}$; $a_2'', \ldots, a_2^{(\sigma-1)}$; and $a_3', \ldots, a_3^{(\sigma-2)}$ only, and are therefore identically satisfied by *any* function of a_1, a_1', a_1'', a_2, a_2', a_3. There are altogether $\sigma + 1$ equations, which form a complete system in $(\sigma + 1) + \sigma + (\sigma - 1)$ variables, and hence there are $2\sigma - 1$ solutions. Writing only those terms of this system which involve differential coefficients with regard to a_1, a_1', a_1'', a_2 and a_3, we have

$$\begin{aligned}
0 &= \frac{\partial f}{\partial a_1} + 2a_1 \frac{\partial f}{\partial a_2} + 3a_2 \frac{\partial f}{\partial a_3}\,, \\
0 &= \frac{\partial f}{\partial a_2} + 3a_1 \frac{\partial f}{\partial a_3} + \frac{\partial f}{\partial a_1'}\,, \\
0 &= \frac{\partial f}{\partial a_3} + \frac{\partial f}{\partial a_1''}\,.
\end{aligned}$$

These three equations in five variables form a complete system, with the two solutions,

$$I_2 = a_2 - a_1^2 - a_1' \quad \text{and} \quad I_3 = a_3 - 3a_2 a_1 + 2a_1^3 - a_1''.$$

I_3 is Cockle's cubicriticoid. Then

$$\frac{dI_2}{dx}, \frac{d^2 I_2}{dx^2}, \ldots\ldots, \frac{d^{\sigma-1} I_2}{dx^{\sigma-1}}; \frac{dI_3}{dx}, \frac{d^2 I_3}{dx^2}, \ldots\ldots, \frac{d^{\sigma-2} I_3}{dx^{\sigma-2}},$$

are the remaining solutions of the complete system, $2\sigma - 3$ in number. There are thus two, and only two, essential invariants involving only a_1, a_2, a_3 and their derivatives.

§8. Proceeding in the same manner, we find that there are three, and only three, essential invariants involving only $a_1, \ldots\ldots, a_4$ and their derivatives. They are the three solutions of the following complete system:

$$\begin{aligned}
0 &= \frac{\partial f}{\partial a_1} + 2a_1 \frac{\partial f}{\partial a_2} + 3a_2 \frac{\partial f}{\partial a_3} + 4a_3 \frac{\partial f}{\partial a_4}\,, \\
0 &= \frac{\partial f}{\partial a_2} + 3a_1 \frac{\partial f}{\partial a_3} + 6a_2 \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1'}\,, \\
0 &= \frac{\partial f}{\partial a_3} + 4a_1 \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1''}\,, \\
0 &= \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1'''}\,.
\end{aligned}$$

I_2 and I_3 are solutions. Call the third solution I_4. Neglecting the terms in a_1' and a_1'' in the second and third equations respectively, we find that we have still a complete system of equations with *one* solution, which must be I_4, since it contains neither a_1' nor a_1''. We have therefore to solve the equations

$$\begin{aligned}
0 &= \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + 4a_3\frac{\partial f}{\partial a_4}, \\
0 &= \frac{\partial f}{\partial a_2} + 3a_1\frac{\partial f}{\partial a_3} + 6a_2\frac{\partial f}{\partial a_4}, \\
0 &= \frac{\partial f}{\partial a_3} + 4a_1\frac{\partial f}{\partial a_4}, \\
0 &= \frac{\partial f}{\partial a_4} + \frac{\partial f}{\partial a_1''}.
\end{aligned}$$

The ordinary method of solving such a system gives as the one solution of this system,

$$I_4 = a_4 - 4a_3 a_1 - 3a_2^2 + 12a_2 a_1^2 - 6a_1^4 - a_1'''.$$

This is Cockle's quarticriticoid.

§9. By exactly similar reasoning we see that the "criticoid" of the k^{th} "degree" is the one solution of the following complete system:

$$\left.\begin{aligned}
0 &= \frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + 3a_2\frac{\partial f}{\partial a_3} + \ldots . + la_{l-1}\frac{\partial f}{\partial a_l} + \ldots . + ka_{k-1}\frac{\partial f}{\partial a_k}, \\
0 &= \frac{\partial f}{\partial a_2} + 3a_1\frac{\partial f}{\partial a_3} + \ldots . + \frac{l(l-1)}{2}a_{l-2}\frac{\partial f}{\partial a_l} + \ldots . + \frac{k(k-1)}{2}a_{k-2}\frac{\partial f}{\partial a_k}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \ldots . + \frac{l!}{(l-r)!\,r!}a_{l-r}\frac{\partial f}{\partial a_l} + \ldots . + \frac{k!}{(k-r)!\,r!}a_{k-r}\frac{\partial f}{\partial a_k}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial f}{\partial a_{k-1}} + ka_1\frac{\partial f}{\partial a_k}, \\
0 &= \frac{\partial f}{\partial a_k} + \frac{\partial f}{\partial a_1^{(k-1)}}.
\end{aligned}\right\} \quad (7)$$

From (4) we see that the r^{th} $(r = 1, 2, \ldots . , k-1)$ equation of this system, being the coefficient of $\phi^{(r)}$ in the extended transformation, has the form

$$0 = \sum_r^k \frac{l!}{(l-r)!\,r!}a_{l-r}\frac{\partial f}{\partial a_l}, \qquad (a_0 = 1) \qquad (8)$$

The solution, I_k, of this system (7) will not be the same as the B_k given earlier, for this solution contains only $a_1, \ldots . , a_k, a_1^{(k-1)}$, while B_k involves in addition to these all the derivatives $a_1', \dots, a_1^{(k-2)}$. The two sets of invariants $B_2, \dots, B_k$ and $I_2, \dots, I_k$ are of course *functionally* equivalent.

If we assign to a_k the *weight* k and to $a_k^{(s)}$ the weight $(k+s)$, then I_2 is isobaric of weight 2, I_3 isobaric of weight 3, and I_4 isobaric of weight 4. We might therefore from analogy assume that I_k is an integral rational function, isobaric of weight k; thus:

$$I_k = a_k - a_1^{(k-1)} + c_{11}a_1a_{k-1} + (c_{12}a_1^2 + c_{22}a_2)\,a_{k-2} \\ + (c_{13}a_1^3 + c_{23}a_2a_1 + c_{33}a_3)\,a_{k-3} + \dots .$$

On substituting this value in the $(k-1)$th equation of the system (7) we find that $c_{11} = -k$. Then the $(k-2)$th equation gives $c_{12} = k(k-1)$, $c_{22} = -\frac{k(k-1)}{2}$, the $(k-3)$th equation will give c_{13}, c_{23}, c_{33}, and so on, the first equation finally giving the coefficient of a_1^k. In the series assumed for I_k no parenthesis is to contain a's of subscript greater than the subscript of the factor into which it is multiplied, otherwise terms will be repeated. We see thus that if I_k be assumed of this form, that values of c can be determined so that it is a solution of all of the equations. Therefore I_k is really isobaric of weight k, and its value may be found in this way by algebraic operations, that is, without integration.

§10. Having now the increments of $y^{(k)}$ and $a_k^{(s)}$ (see (3) and (6)), the work of finding the covariants is short. For the equation of the third order, we have from (3) and (6), (the factor δt being omitted):

$$\delta y = \phi y,\quad \delta y' = \phi' y + \phi y',\quad \delta y'' = \phi'' y + 2\phi' y' + \phi y'',$$
$$\delta a_1 = -\phi',\quad \delta a_1' = -\phi'',\quad \delta a_1'' = -\phi''',\quad \delta a_2 = -(\phi'' + 2a_1\phi'),$$
$$\delta a_2' = -(\phi''' + 2a_1\phi'' + 2a_1'\phi'),\quad \delta a_3 = -(\phi''' + 3a_1\phi'' + 3a_2\phi'),$$

Forming the extended transformation, and, as before, equating the coefficients of the different derivatives of ϕ to zero, we get

$$0 = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y''\frac{\partial f}{\partial y''},$$
$$0 = y\frac{\partial f}{\partial y'} + 2y'\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial a_1} - 2a_1\frac{\partial f}{\partial a_2} - 2a_1'\frac{\partial f}{\partial a_2'} - 3a_2\frac{\partial f}{\partial a_3}$$
$$0 = y\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial a_1'} - \frac{\partial f}{\partial a_2} - 2a_1\frac{\partial f}{\partial a_2'} - 3a_1\frac{\partial f}{\partial a_3},$$
$$0 = \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + \frac{\partial f}{\partial a_3}.$$

This complete system of four equations in nine variables has five solutions, for the equations are evidently independent. Three of these solutions are I_2, I_3 and $\frac{dI_2}{dx}$, and are free from y. The remaining two involve y or its derivatives, and are absolute covariants. They are:

$$\frac{y' + a_1 y}{y} \quad \text{and} \quad \frac{y'' + 2a_1 y' + a_2 y}{y}$$

These agree with the relative covariants given by Cockle, for since y is a relative covariant, the numerators of both of these absolute covariants must be relative covariants. Comparing Cockle's operative symbol, Δ, with the above equations, we see that it is equivalent to the second of them. For the *relative* invariants the zero in the left-hand member of the first of the above equations must be replaced by f (see §39). Cockle therefore had found only *one* of the four partial differential equations which his covariant satisfies.

§11. For the covariants of the general equation of the n^{th} order, we have, from (3) and (6), as the extended transformation,

$$\sum_{0}^{w}{}_{k}\sum_{0}^{k}{}_{m}\frac{k!}{(k-m)!\,m!}\phi^{(m)}y^{(k-m)}\frac{\partial f}{\partial y^{(k)}}$$
$$-\sum_{1}^{n}{}_{k}\sum_{0}^{w-k}{}_{s}\sum_{1}^{k}{}_{i}\sum_{i}^{s+i}{}_{m}\frac{s!\,k!\,\phi^{(m)}a_{k-i}^{(s+i-m)}}{(k-i)!\,i!\,(s+i-m)!\,(m-i)!}\frac{\partial f}{\partial a_k^{(s)}},$$

where $w \geqq n$ is an integer, and the transformation is so far extended that terms of weight not greater than w enter. On changing the order of summation, this becomes

$$\sum_{0}^{w}{}_{m}\phi^{(m)}\sum_{m}^{w}{}_{k}\frac{k!}{(k-m)!\,m!}y^{(k-m)}\frac{\partial f}{\partial y^{(k)}}$$
$$-\sum_{1}^{w}{}_{m}\phi^{(m)}\sum_{1}^{n}{}_{k}\sum_{\left\{\substack{m-k\\0}\right.}^{w-k}{}_{s}\sum_{\left\{\substack{m-s\\1}\right.}^{\left\{\substack{k\\m}\right.}{}_{i}\frac{s!\,k!\,a_{k-i}^{(s+i-m)}}{(k-i)!\,i!\,(s+i-m)!\,(m-i)!}\frac{\partial f}{\partial a_k^{(s)}}.$$

In this expression, when two upper limits are written, the smaller is chosen, and when two lower limits are written the larger one is to be used. Equating to zero the coefficient of $\phi^{(m)}$, we obtain the following complete system of partial differential equations:

$$\left.\begin{aligned} 0 &= \sum_{0}^{w} {}_k\, y^{(k)} \frac{\partial f}{\partial y^{(k)}}, \\ 0 &= \sum_{m}^{w} {}_k\, \frac{k!}{(k-m)!\; m!} y^{(k-m)} \frac{\partial f}{\partial y^{(k)}} \\ &\quad - \sum_{1}^{n} {}_k \sum_{\left\{\begin{smallmatrix} m-k \\ 0 \end{smallmatrix}\right.}^{w-k} {}_s \sum_{\left\{\begin{smallmatrix} m-s \\ 0 \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} k \\ m \end{smallmatrix}\right.} {}_i\, \frac{s!\; k!\; a_{k-i}^{(s+i-m)}}{(k-i)!\; i!\; (s+i-m)!\; (m-i)!} \frac{\partial f}{\partial a_k^{(s)}}\,. \\ & \qquad\qquad (m = 1, 2, \ldots\ldots, w) \end{aligned}\right\} \quad (9)$$

These $w + 1$ equations are linearly independent, for the first determinant of the matrix of the coefficients has the value $y^{w+1} \neq 0$. The number of variables is

$$w + 1 + \frac{w(w+1) - (w-n)(w-n+1)}{2},$$

so that the number of solutions is $nw - \frac{1}{2}(n^2 - n)$. Consider in particular the the case $w = n$. The number of absolute covariants is then $\frac{1}{2}n(n+1)$. Among this number we have the invariants $I_2, I_3, \ldots\ldots, I_n$, and their derivatives up to those of weight n. The total number of these solutions is:

$$(n-1) + (n-2) + \ldots\ldots + 2 + 1 = \tfrac{1}{2}n(n-1).$$

There are consequently exactly $\frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) = n$ other solutions, and these are

$$\frac{y' + a_1 y}{y}, \quad \frac{y'' + 2a_1 y' + a_2 y}{y}, \ldots\ldots, \quad \frac{y^{(n)} + na_1 y^{(n-1)} + \ldots\ldots + a_n y}{y}.$$

We see that these quantities are solutions as follows: Let us use Cockle's notation

$$y_r = a_0 y^{(r)} + ra_1 y^{(r-1)} + \ldots\ldots + a_r y = \sum_{0}^{r} {}_j\, \frac{r!}{(r-j)!\; j!} a_j y^{(r-j)}; \qquad (a_0 = 1).$$

We have so determined $\delta y^{(k)}$ and $\delta a_k^{(s)}$ that $y_n = 0$ is an invariant differential equation. But the expressions (3) and (6) for $\delta y^{(k)}$ and $\delta a_k^{(s)}$ do not involve n, the order of the equation, and would be the same if determined from any other equation $y_r = 0$. Therefore the equation $y_r = 0$, $(r = 1, 2, \ldots, n)$ must be an invariant differential equation for the transformation in question. That is,

$$y_r \equiv \rho v_r,$$

when the substitution $y = u . v$ is made. The factor ρ may be determined by considering the highest differential coefficient

$$y^{(r)} = uv^{(r)} + \ldots.$$

Hence

$$y_r = \rho v_r = uv_r,$$

$$\frac{y_r}{y} = \frac{v_r}{v},$$

and $\frac{y_r}{y}$ $(r = 1, 2, \ldots, n)$ are absolute covariants.

This same fact may also be easily proved by direct substitution in (9). When we substitute

$$y_r = \sum_{0}^{r}{}_{j} \frac{r\,!}{(r-j)\,!\; j\,!} a_j y^{(r-j)}$$

in any one of the last $w = n$ equations (9) we have only to retain in the second summation those terms which involve a_j; that is, we may write $s = 0$, $i = m$, so that the equations become

$$0 = \sum_{m}^{n}{}_{k} \frac{k\,!}{(k-m)\,!\; m\,!} y^{(k-m)} \frac{\partial f}{\partial y^k} - \sum_{m}^{n}{}_{k} \frac{k\,!}{(k-m)\,!\; m\,!} a_{k-m} \frac{\partial f}{\partial a_k} \equiv A_m f.$$

$$(m = 1, 2, \ldots, n)$$

We have

$$\frac{\partial y_r}{\partial y^{(k)}} = \frac{r\,!}{(r-k)\,!\; k\,!} a_{r-k}, \quad \frac{\partial y_r}{\partial a_k} = \frac{r\,!}{(r-k)\,!\; k\,!} y^{(r-k)}, \qquad (0 \leqq k \leqq r)$$

and

$$\frac{\partial y_r}{\partial y^{(k)}} = 0, \quad \frac{\partial y_r}{\partial a_k} = 0 \text{ when } k > r.$$

Hence

$$\begin{aligned} A_m y_r &= \sum_{m}^{r}{}_{k} \frac{k\,!}{(k-m)\,!\; m\,!} \, \frac{r\,!}{(r-k)\,!\; k\,!} \left[y^{(k-m)} a_{r-k} - y^{(r-k)} a_{k-m} \right] \\ &= \sum_{m}^{r}{}_{k} \frac{r\,!}{(k-m)\,!\; (r-k)\,!\; m\,!} y^{(k-m)} a_{r-k} - \sum_{m}^{r}{}_{j} \frac{r\,!}{(j-m)\,!\; (r-j)\,!\; m\,!} y^{(j-m)} a_{r-j} \\ &= 0. \end{aligned}$$

That is, y_r is a solution of the last n equations, and the first equation tells us that a solution must be homogeneous in the y's. Therefore y_r/y is a solution of the system (9). When $w > n$ the remaining solutions are obtained by differentiating those already found with regard to x.

Cockle had therefore in 1865 indicated the complete system of invariants and covariants of a homogeneous linear differential equation of the n^{th} order, for a transformation of the dependent variable of the form $y = v \,.\, u(x)$. *His* system contained more than the requisite number of solutions, for we have just seen that the solutions $\frac{d^p y_q}{dx^p}$ are not needed.

§12. Let us now take up the case in which the independent variable alone is transformed. Then

$$x_1 = \chi(x), \quad y_1 = y,$$

where x_1, y_1 are the transformed variables, and χ is an arbitrary function of x. These transformations form an infinite continuous group defined by differential equations. We get the infinitesimal transformation by writing $\chi = x + \xi(x)\,\delta t$. Then

$$\delta x = \xi(x)\,\delta t, \quad \delta y = 0,$$

and the symbol of the transformation is

$$\xi(x)\frac{\partial f}{\partial x}.$$

The following work is not made perfectly general, but is carried only far enough to get the covariant given by Cockle in 1875:

$$\delta y' = \frac{d\delta y}{dx} - y'\frac{d\delta x}{dx} = -\xi' y'\,\delta t.$$

$$\delta y^{(k)} = \frac{d\delta y^{(k-1)}}{dx} - y^{(k)}\frac{d\delta x}{dx} = -y^{(k)}\xi'\,\delta t + \frac{d}{dx}\left(\delta y^{(k-1)}\right),$$

$$= -\delta t \,.\left[y^{(k)}\xi' + \frac{d}{dx}\left(y^{(k-1)}\xi'\right) + \frac{d^2}{dx^2}\left(y^{(k-2)}\xi'\right) + \ldots . + \frac{d^{k-1}}{dx^{k-1}}\left(y'\xi'\right)\right]$$

$$= -\delta t \sum_{0}^{k-1}{}_{s} \frac{d^s}{dx^s}\left(y^{(k-s)}\xi'\right) = -\delta t \sum_{0}^{k-1}{}_{s} \sum_{0}^{s}{}_{m} \frac{s\,!}{(s-m)\,!\; m\,!}\, y^{(k-m)}\xi^{(m+1)}$$

$$= -\sum_{0}^{k-1}{}_{m}\, y^{(k-m)}\xi^{(m+1)} \sum_{m}^{k-1}{}_{s} \frac{s\,!}{(s-m)\,!\; m\,!}.$$

Since $\sum_{m}^{k-1}{}_{s} \frac{s!}{(s-m)!\,m!} = \frac{k!}{(k-m-1)!\,(m+1)!}$, this becomes

$$\delta y^{(k)} = -\,\delta t \sum_{0}^{k-1}{}_{m} \frac{k!}{(k-m-1)!\,(m+1)!} y^{(k-m)}\xi^{(m+1)}. \tag{10}$$

As in the case for change of the dependent variable, we have

$$\delta y^{(n)} + na_1\,\delta y^{(n-1)} + \ldots. + a_n\,\delta y + ny^{(n-1)}\,\delta a_1 + \ldots. + y\,\delta a_n = 0.$$

On substituting the values of $\delta y^{(n)}$, $\delta y^{(n-1)}$, $\delta y^{(n-2)}$ as given by (10), this becomes

$$0 = -n\xi' y^{(n)} \begin{array}{r|r|l} -\frac{n(n-1)}{2}\xi'' & y^{(n-1} - \frac{n(n-1)(n-2)}{3!}\xi''' & y^{(n-2)} + \ldots. \\ -n(n-1)\,a_1\xi' & -\frac{n(n-1)(n-2)}{2}a_1\xi'' & \\ +\,n\delta a_1 & -\frac{n(n-1)(n-2)}{2}a_2\xi' & \\ & +\frac{n(n-1)}{2}\delta a_2 & \end{array}$$

Eliminate the term involving $y^{(n)}$ by means of (2) and equate the coefficients of $y^{(n-1)}$ and $y^{(n-2)}$ to zero. This gives

$$\delta a_1 = \left(\frac{n-1}{2}\xi'' - a_1\xi'\right)\delta t,$$

$$\delta a_2 = \left(\frac{n-2}{3}\xi''' + (n-2)\,a_1\xi'' - 2a_2\xi'\right)\delta t,$$

$$\delta a_1' = \delta\frac{da_1}{dx} = \frac{d}{dx}\delta a_1 - a_1'\xi'\delta t = \left(\frac{n-1}{2}\xi''' - a_1\xi'' - 2a_1'\xi'\right)\delta t.$$

The extended transformation is then

$$\xi\frac{\partial f}{\partial x} - \xi' y'\frac{\partial f}{\partial y'} + \left(\frac{n-1}{2}\xi'' - a_1\xi'\right)\frac{\partial f}{\partial a_1}$$
$$+ \left(\frac{n-2}{3}\xi''' + (n-2)\,a_1\xi'' - 2a_2\xi'\right)\frac{\partial f}{\partial a_2}$$
$$+ \left(\frac{n-1}{2}\xi''' - a_1\xi'' - 2a_1'\xi'\right)\frac{\partial f}{\partial a_1'}\,.$$

Equating the coefficients of the different derivatives of ξ to zero, we have

$$\begin{aligned} 0 &= y' \frac{\partial f}{\partial y'} + a_1 \frac{\partial f}{\partial a_1} + 2a_2 \frac{\partial f}{\partial a_2} + 2a_1' \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{n-1}{2} \frac{\partial f}{\partial a_1} + (n-2) a_1 \frac{\partial f}{\partial a_2} - a_1 \frac{\partial f}{\partial a_1'}, \\ 0 &= \frac{n-2}{3} \frac{\partial f}{\partial a_2} + \frac{n-1}{2} \frac{\partial f}{\partial a_1'}. \end{aligned}$$

By the ordinary methods the one solution of this complete system of three equations in four variables is found to be

$$\left[a_1' + \frac{3n-1}{2n-2} a_1^2 - \frac{3(n-1)}{2(n-2)} a_2\right] \frac{1}{y'^2}, \qquad n > 2.$$

This is Cockle's "differential quadricriticoid." Other invariants found by extending the group further are not here given because they are simply the solutions of equations to be given later. We have now found all of Cockle's invariants by Lie's methods, and proceed to the general case.

CHAPTER 3.

Invariants of the General Homogeneous Linear Differential Equation in Two Variables, for Transformation of Both Variables.

§13. It is known* that the most general point transformation which transforms the equation

$$a_0 y^{(n)} + n a_1 y^{(n-1)} + \dots + a_n y = \sum_0^n \frac{n!}{(n-s)!\, s!} a_s y^{(n-s)} = 0, \quad (a_0 = 1) \tag{2}$$

into an equation of the same form is, when $n > 1$, one of the form

$$x_1 = \chi(x), \quad y_1 = y \Psi(x). \tag{11}$$

These transformations (11) form an infinite continuous group defined by differential equations. Lie's theory may therefore be used in finding all the invariants of the equation (2) for the transformations (11).

§14. Let us consider the effect of a finite transformation before taking up the infinitesimal. The following treatment follows closely that for finding the factor by which a relative differential invariant in x, y, y', y'', ... is multiplied when subjected to a point transformation. It is thought best to make this treatment only general enough for the purposes of this paper, and not to give the factor in determinant form, as might be done. We have

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* P. Stäckel, Journal für Math., CXI, pp. 290–302.

$$\begin{aligned}
y_1 &= y\Psi(x), \quad x_1 = \chi(x), \\
y_1' &= \frac{\Psi}{\chi'} y' + \frac{\Psi'}{\chi'} y, \\
y_1'' &= \frac{\Psi}{\chi'^2} y'' + \frac{2'\Psi\chi' - \Psi\chi''}{\chi'^3} y' + \frac{\chi'\Psi'' - \chi''\Psi'}{\chi'^3} y, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_1^{(\mu)} &= \frac{\Psi}{\chi'^{\mu}} y^{(\mu)} + \sum_{0}^{\mu-1}{}_{s}\, \beta_{\mu s}(x) \,.\, y^{(s)}.
\end{aligned} \tag{12}$$

When these values are substituted in

$$y_1^{(n)} + nb_1 y_1^{(n-1)} + \ldots . + b_n y_1 = 0$$

we must get a multiple of (2), for (2) must be an invariant equation, and the substitution (12) is linear in the $y^{(k)}$'s; therefore

$$y_1^{(n)} + nb_1 y_1^{(n-1)} + \ldots . + b_n y_1 \equiv \frac{\Psi}{\chi'^n}\left[y^{(n)} + na_1 y^{(n-1)} + \ldots . + a_n y\right],$$

$$\sum_{0}^{n}{}_{s} \frac{n!}{(n-s)!\; s!} b_s y_1^{(n-s)} \equiv \frac{\Psi}{\chi'^n} \sum_{0}^{n}{}_{s} \frac{n!}{(n-s)!\; s!} a_s y^{(n-s)}. \tag{13}$$

Substitution of (12) in (13) and comparison of the coefficients of $y^{(n-j)}$ gives

$$a_j = \chi'^j b_j + \frac{\chi'^n}{\Psi} \sum_{0}^{j-1}{}_{k} \frac{(n-j)!\; j!}{(n-k)!\; k!} b_k \beta_{n-j,\, n-k}. \quad (j = 1, 2, \ldots . , n). \tag{14}$$

Solving these n equations for b_j $(j = 1, 2, \ldots . , n)$ gives

$$b_j = \frac{1}{\chi'^j} a_j + \sum_{0}^{j-1}{}_{s} \gamma_{js}(x) \,.\, a_s, \quad (j = 1, 2, \ldots . , n) \tag{15}$$

Differentiating (15) k times with regard to x gives

$$b_j^{(k)} = \frac{1}{\chi'^{j+k}} a_j^{(k)} + \sum_{0}^{j}{}_{s} \sum_{0}^{k}{}_{i} \vartheta_{jksi}(x) \,.\, a_s^{(i)}, \text{ where } \vartheta_{jkjk} = 0. \tag{16}$$

$$(j = 1, 2, \ldots . , n, \; k = 0, 1, 2, \ldots .)$$

The formula (16) includes (15), and (16) and (12) together with (11) are the finite equations of the extended group. Since (12) and (16) are linear substitutions, any integral rational function in $y^{(\mu)}$ and $a_j^{(k)}$ will be transformed into an integral rational function whose degree taken in either set of variables alone, or in both together, will be the same as that of the original function.

§15. Now make the transformation

$$x_1 = x, \quad y_1 = Cy,$$

which is included in (11). Then

$$y_1^{(\mu)} = Cy^{(\mu)}, \quad \text{and} \quad b_j^{(k)} = a_j^{(k)}.$$

Therefore any differential invariant

$$I(x, y, y' \dots . y^{(\nu)}; \quad a_1 \dots . a_n, \dots . a_j^{(k)} \dots .)$$

must be homogeneous in $y, y', \dots . , y^{(\nu)}$ of degree zero, and any invariant differential equation must be homogeneous in $y, y', \dots . , y^{(\nu)}$

Next make the transformation,

$$y_1 = y, \quad x_1 = Cx,$$

which is also included in (11). Then

$$y_1^{(\mu)} = C^{-\mu} y^{(\mu)}, \quad b_j^{(k)} = C^{-(j+k)} a_j^{(k)}.$$

Let us assign to $[y^{(\mu)}]^\nu$ the weight $\mu\nu$, and to $[a_j^{(k)}]^l$ the weight $(j+k)\,l$. Then these equations show that any invariant must be isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s of weight zero, and that any invariant differential equation must be isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s. Hence we have the theorem:

Theorem I.

Any absolute invariant of the linear differential equation (2) for the group of transformations (11) must be homogeneous in the $y^{(\mu)}$'s of degree zero and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s of weight zero. An invariant equation (or relative invariant) must be homogeneous in the $y^{(\mu)}$'s and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s.

This theorem must hold for every subgroup of (11) which contains the two transformations

$$\begin{cases} y_1 = Cy, \\ x_1 = x \ , \end{cases} \quad \begin{cases} y_1 = y \ , \\ x_1 = Cx. \end{cases}$$

§16. Let us consider an integral rational function which is a relative invariant. Then by Theorem I it is homogeneous in the $y^{(\mu)}$'s, say of degree λ, and isobaric in the $y^{(\mu)}$'s and $a_j^{(k)}$'s, say of weight w. Represent this by

$$\Omega^{(\lambda,\, w)} = 0 .$$

This equation is invariant. $\Omega^{(\lambda,\, w)}$ must be composed of individual terms, each of which is of degree λ in the $y^{(\mu)}$'s and of weight w in the $y^{(\mu)}$'s and $a_j^{(k)}$'s. The formulæ (12) and (16) show that any such term is to be replaced by the same term multiplied by Ψ^{λ}/χ'^{w}, plus a function of weight *less* than w. Then

$$\Omega_1^{(\lambda,\, w)} \equiv \frac{\psi^{\lambda}}{\chi'^{w}}\, \Omega^{(\lambda,\, w)} + \Theta ,$$

where Θ is an integral rational function, none of whose terms is of weight as great as w. The left-hand member of this identity vanishes when we place $\Omega^{(\lambda,\, w)} = 0$, for this is an invariant equation. Hence Θ is identically zero, for a function of weight less than w cannot vanish by means of one of weight w. That is,

$$\Omega_1^{(\lambda,\, w)} \equiv \frac{\Psi^{\lambda}}{\chi'^{w}}\, \Omega^{(\lambda,\, w)} . \tag{17}$$

This formula (17) shows how an invariant integral rational equation is transformed. The same is true if $\Omega^{(\lambda,\, w)}$ is any root of an integral rational function, extensions of the definition of degree and weight to such a function being made. These are the only functions which arise in the present article. If an invariant irrational algebraic equation were given, it could be rationalized, and then (17) would give the factor by which the invariant is multiplied when a transformation (11) is made.

Theorem II.—Whenever three relative invariants are known, an absolute invariant may be constructed by algebraic processes.

Let $\Omega^{(\lambda_1,\, w_1)}$, $\Omega^{(\lambda_2,\, w_2)}$ $\Omega^{(\lambda_3,\, w_3)}$ be the three relative invariants. Then (17) shows that

$$[\Omega^{(\lambda_1,\, w_1)}]^{\mu_1}\, [\Omega^{(\lambda_2,\, w_2)}]^{\mu_2}\, [\Omega^{(\lambda_3,\, w_3)}]^{\mu_3} \tag{18}$$

is an absolute invariant, if μ_1, μ_2, μ_3 be determined from the equations

$$\begin{aligned} \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 &= 0 , \\ w_1\mu_1 + w_2\mu_2 + w_3\mu_3 &= 0 , \end{aligned}$$

that is, if

$$\mu_1 : \mu_2 : \mu_3 = \begin{vmatrix} \lambda_2 & \lambda_3 \\ w_2 & w_3 \end{vmatrix} : \begin{vmatrix} \lambda_3 & \lambda_1 \\ w_3 & w_1 \end{vmatrix} : \begin{vmatrix} \lambda_1 & \lambda_2 \\ w_1 & w_2 \end{vmatrix} . \tag{19}$$

This formula always gives possible values of $\mu_1 : \mu_2 : \mu_3$, except when

$$\lambda_1 : \lambda_2 : \lambda_3 = w_1 : w_2 : w_3,$$

in which case it is necessary to know only two different invariant equations. This case occurs when two of the given relative invariants are free from y and its derivatives; it is then not necessary to know a third. No one of the Ω's may be an absolute invariant if (18) is to give a *new* absolute invariant.

This theorem is the exact analogon of a well-known theorem of Lie concerning invariant differential equations of curves in a plane.

Theorem III.—If an absolute invariant be a rational algebraic function, both its numerator and denominator are relative invariants, of the same degree and weight. Let $I = U/V$ be an absolute invariant, where U and V are integral rational functions, which by (12) and (16) transform into integral rational functions, say U_1 and V_1 respectively.

$$\frac{U}{V} = k$$

is an invariant equation, and this invariance is not destroyed if we clear of fractions.

$$U - kV = 0$$

is therefore an invariant integral rational equation, and must be homogeneous in the y's and isobaric (Theorem I). Say the degree in the y's is λ and the weight w. Then by (17)

$$U_1 - kV_1 = \frac{\Psi^\lambda}{\chi'^w}(U - kV).$$

This equation is true for all values of k. Hence

$$U_1 = \frac{\Psi^\lambda}{\chi'^w} U, \quad V_1 = \frac{\Psi^\lambda}{\chi'^w} V. \qquad \text{Q. E. D.}$$

Theorem IV.—The total derivative with regard to x of an absolute invariant is a relative invariant of next higher order.

If I be an absolute invariant, then

$$I_1 = I,$$

and this is an identity when we make use of (11), (12) and (16). Hence we may differentiate totally with regard to x. This gives

$$\frac{dI_1}{dx_1} = \frac{1}{\chi'}\,\frac{dI}{dx},$$

which shows that $\frac{dI}{dx}$ is a relative invariant, homogeneous of degree zero in the y's and isobaric of weight unity. Its order in both y and the a's is evidently one greater than that of I.

As a corollary, we see that if we have two absolute invariants, I_1 and I_2, that then

$$\frac{dI_2}{dx}\Big/\frac{dI_1}{dx}, \quad \text{or} \quad \frac{dI_2}{dI_1}$$

is an absolute invariant, and

$$\frac{dI_2}{dI_1},\ \frac{d^2I_2}{dI_1^2},\ \frac{d^3I_2}{dI_1^3},\ \ldots . \tag{20}$$

is an infinite series of absolute invariants in ascending order. For transformation of the dependent variable alone, we may write $I_1 \equiv x$. This justifies the method used at the end of §6.

§17. The preceding theorems enable us to derive an infinite series of absolute invariants in ascending order from *two known relative invariants.* We always know a third relative invariant, namely, y, for we have $y_1 = \Psi y$. Then applying Theorem II gives an absolute invariant, and IV gives a new relative invariant of next higher order. Using II again yields a second absolute invariant, and this alternating process may be continued, or (20) may be used, and we thus get an infinite series of absolute invariants in ascending order.

§18. If a *single absolute invariant*, say

$$I = \frac{U^{(\lambda,\,w)}}{V^{(\lambda,\,w)}},$$

be known, Theorems II, III, IV, together with the foregoing remarks, show that

$$J = \left[\frac{y^{\lambda}}{V^{(\lambda,\,w)}}\right]^{\frac{1}{w}} \frac{dI}{dx} \tag{21}$$

is an absolute invariant of order one greater than that of I. A repetition of the process gives an infinite series of absolute invariants from a single absolute invariant.

§19. For subgroups of the general group (11) it is often possible to find an infinite series of absolute invariants by differentiation processes from a *single* relative invariant. Thus if

$$\Psi = C\chi'^{\nu},$$

where C is an arbitrary and ν a known constant, we have a subgroup of (11). Suppose that for this subgroup we know a relative invariant U of degree λ in the y's and isobaric of weight w. Then we have

$$U_1 = \frac{\Psi^{\lambda}}{\chi'^{w}}\, U = C^{\lambda}\chi'^{\lambda\nu - w}\, U; \quad y_1 = C\chi'^{\nu} y;$$

$$U_1' = C^{\lambda}\chi'^{\lambda\nu - w - 1}\, U' + C^{\lambda}(\lambda\nu - w)\,\chi'^{\lambda\nu - w - 2}\chi'' U; \quad y_1' = C\chi'^{\nu - 1} y' + C\nu\chi'^{\nu - 2}\chi'' y.$$

The elimination of χ', χ'', and C between these four equations shows that

$$\frac{\nu U'y - (\lambda\nu - w)\, Uy'}{U^{\frac{w+1}{w}}\, y^{\frac{w-\lambda}{w}}} \tag{22}$$

is an absolute invariant for the subgroup

$$x_1 = \chi(x), \quad y_1 = Cy\chi'^{\nu},$$

when U is a relative invariant, of degree λ in the y's, and isobaric of weight w.

For other subgroups other methods of deriving new invariants from known invariants by differentiation can be developed (cf. Forsyth's quadriderivative and Jacobian processes)*, but as we have now all the methods we need, we shall not go into this. (Cf. §51.)

§20. With these general remarks on the methods of finding new invariants from known invariants, we return to the actual process of finding the invariants by the method of infinitesimal transformations.

The symbol of the infinitesimal transformations of the group (11) is

$$Xf \equiv \xi(x)\frac{\partial f}{\partial x} + \phi(x)\, y\frac{\partial f}{\partial y}, \tag{23}$$

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*Phil. Trans., I, 1888, pp. 407-418.

where ξ and ϕ are arbitrary functions of x alone. This transformation must be extended by taking into account the variation of $y^{(k)}$ and $a_j^{(k)}$. We have

$$\begin{aligned}\delta y^{(k)} &= \delta \frac{dy^{(k-1)}}{dx} = -y^{(k)}\xi'\,\delta t + \frac{d}{dx}(\delta y^{(k-1)}) \\ &= -\left[y^{(k)}\xi' + \frac{d}{dx}(y^{(k-1)}\xi') + \dots + \frac{d^{k-1}}{dx^{k-1}}(y'\xi')\right]\delta t + \frac{d^k}{dx^k}(\delta y) \\ &= \left[-\sum_{s=0}^{k-1} \frac{d^s}{dx^s}(y^{(k-s)}\xi') + \frac{d^k}{dx^k}(\phi y)\right]\delta t.\end{aligned}$$

Making use of Euler's theorem for the differentiation of a product, and of the formula

$$\sum_{s=m}^{k-1} \frac{s!}{(s-m)!\,m!} = \frac{k!}{(k-m-1)!\,(m+1)!},$$

we find, in the same way that (10) was found,

$$\delta y^{(k)} = \sum_{m=0}^{k} \frac{k!}{(k-m)!\,(m+1)!}\left[(m+1)\phi^{(m)} - (k-m)\xi^{(m+)}\right] y^{(k-m)}\,\delta t. \tag{24}$$

§21. The variation of (2) gives

$$\sum_{s=1}^{n} \frac{n!}{(n-s)!\,s!}\, y^{(n-s)}\,\delta a_s + \sum_{s=0}^{n} \frac{n!}{(n-s)!\,s!}\, a_s\,\delta y^{(n-s)} = 0.$$

On substituting in this equation the value of $\delta y^{(n-s)}$ from (24) we get

$$\begin{aligned}&\sum_{s=1}^{n} \frac{n!}{(n-s)!\,s!}\, y^{(n-s)}\delta a_s \\ &= -\sum_{s=0}^{n} \frac{n!\,a_s}{(n-s)!\,s!} \sum_{m=0}^{n-s} \frac{(n-s)!\,y^{(n-m-s)}}{(n-m-s)!\,(m+1)!}\left[(m+1)\phi^{(m)} - (n-m-s)\xi^{(m+1)}\right]\delta t \\ &= -\sum_{s=0}^{n}\sum_{j=s}^{n} \frac{n!\,a_s y^{(n-j)}\left[(j-s+1)\phi^{(j-s)} - (n-j)\xi^{(j-s+1)}\right]}{s!\,(n-j)!\,(j-s+1)!}\,\delta t \\ &= -\sum_{j=0}^{n}\sum_{s=0}^{j} \frac{n!\,a_s y^{(n-j)}\left[(j-s+1)\phi^{(j-s)} - (n-j)\xi^{(j-s+1)}\right]}{s!\,(n-j)!\,(j-s+1)!}\,\delta t \\ &= -\sum_{j=1}^{n} \frac{n!\,y^{(n-j)}}{(n-j)!\,j!} \sum_{s=0}^{j-1} \frac{j!\,a_s}{s!\,(j-s+1)!}\left[(j-s+1)\phi^{(j-s)} - (n-j)\xi^{(j-s+1)}\right]\delta t \\ &\quad -\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, a_j y^{(n-j)}\left[\phi - (n-j)\xi'\right]\delta t.\end{aligned}$$

But in consequence of (2), $\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, a_j\, y^{(n-j)}(\phi - n\xi') = 0$. Hence we finally get by equating coefficients of $y^{(n-j)}$,

$$\delta a_j = \left\{ -j a_j \xi' - \sum_{s=0}^{j-1} \frac{j!\; a_s}{(j-s+1)!\; s!} \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] \right\} \delta t. \quad (25)$$

$$(j = 1,\, 2,\, \ldots\ldots,\, n)$$

§22. In extending the group of transformations we also require $\delta a_j^{(k)}$.

$$\delta a_j^{(k)} = \delta \frac{d a_j^{(k-1)}}{dx} = -\xi' a_j^{(k)}\, \delta t + \frac{d}{dx}\left(\delta a_j^{(k-1)}\right)$$

$$= -\left[a_j^{(k)} \xi' + \frac{d}{dx}\left(a_j^{(k-1)} \xi'\right) + \ldots\ldots + \frac{d^{k-1}}{dx^{k-1}}\left(a_j' \xi'\right) \right] \delta t + \frac{d^k}{dx^k}(\delta a_j).$$

The quantity within the bracket is the same as that encountered in finding $\delta y^{(k)}$, a_j replacing y. Hence

$$\delta a_j^{(k)} = -\sum_{m=0}^{k} \frac{k!\; \xi^{(m+1)}}{(k-m-1)!\; (m+1)!}\, a_j^{(k-m)}\, \delta t + \frac{d^k}{dx^k}(\delta a_j)$$

$$= -\sum_{m=0}^{k} \frac{k!\,(k-m)}{(k-m)!\,(m+1)!}\, \xi^{(m+1)}\, a_j^{(k-m)} \delta t - j \delta t \sum_{m=0}^{k} \frac{k!}{(k-m)!\; m!}\, \xi^{(m+1)}\, a_j^{(k-m)}$$

$$- \delta t \,.\, \sum_{s=0}^{j-1} \frac{j!}{(j-s+1)!\; s!}\; \frac{d^k}{dx^k} \left\{ \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] a_s \right\}.$$

Combining the first two summations into one, and changing the letter of summation, gives

$$\delta a_j^{(k)} = -\,\delta t \left[\begin{array}{l} \displaystyle\sum_{s=0}^{k} \frac{k!\; [j(k+1) - s(j-1)]}{s!\; (k-s+1)!}\, a_j^{(s)}\, \xi^{(k-s+1)} \\ \displaystyle + \sum_{s=0}^{j-1} \frac{j!}{s!\; (j-s+1)!}\; \frac{d^k}{dx^k} \left\{ \left[(j-s+1)\,\phi^{(j-s)} - (n-j)\,\xi^{(j-s+1)} \right] a_s \right\} \end{array} \right]. \quad (26)$$

The form (26) is useful for some purposes, but in order to find the equations which give the invariants, the indicated differentiation under the second summation sign must actually be carried out, and then the summations arranged according to $\xi^{(s)}$ and $\phi^{(s)}$. Performing the differentiation gives

$$\frac{\delta a_j^{(k)}}{\delta t} = -\sum_{1}^{k+1}{}_{s}\, \frac{k\,!\; a_j^{(k-s+1)}[k+1+s(j-1)]}{(k-s+1)\,!\; s\,!}\,\xi^{(s)}$$

$$-\sum_{0}^{j-1}{}_{s}\;\sum_{j-s}^{k+j-s}{}_{m}\, \frac{j\,!\; k\,!\; a_s^{(k+j-s-m)}}{(j-s)\,!\; s\,!\; (k+j-s-m)\,!\; (s-j+m)\,!}\,\phi^{(m)}$$

$$+\sum_{0}^{j-1}{}_{s}\;\sum_{j+1-s}^{k+j+1-s}{}_{m}\, \frac{j\,!\; k\,!\; (n-j)\, a_s^{(k+j+1-s-m)}}{(j-s+1)\,!\; s\,!\; (k+j-s-m+1)\,!\; (m+s-j-1)\,!}\,\xi^{(m)}.$$

In the double sums the order of summation must now be changed so that the summation with regard to m comes first.

§23. In changing the order of summation the typical term will remain the same, only the *limits* of summation being changed. The limits of summation are changed as in the following example:

$\sum_{0}^{j-1}{}_{s}\;\sum_{j-s}^{k+j-s}{}_{m}$ shows that $j-s \leqq m \leqq k+j-s$. Hence if m be first chosen, we have for s,

$$j-m \leqq s \leqq k+j-m.$$

s also lies between other limits, namely: $0 \leqq s \leqq j-1$, and sometimes one set and sometimes another must be chosen. The third inequality combined with the first gives

$$1 < m \leqq k+j,$$

$$\therefore \sum_{0}^{j-1}{}_{s}\;\sum_{j-s}^{k+j-s}{}_{m} = \sum_{1}^{k+j}{}_{m}\;\sum_{\text{larger of}\left\{\begin{smallmatrix} j-m \\ 0 \end{smallmatrix}\right.}^{\text{smaller of}\left\{\begin{smallmatrix} k+j-m \\ j-1 \end{smallmatrix}\right.}{}_{s}\,.$$

In all of the following work if two (or more) upper limits are written the smaller is to be chosen, and if two (or more) lower limits are written the larger is to be chosen.

§24. Changing the orders of summation in this way gives

$$\frac{\delta a_j^{(k)}}{\delta t} = -\sum_{s=1}^{k+1} \frac{k\,!\,[k+1+s(j-1)]}{(k-s+1)\,!\;s\,!}\,a^{(k-s+1)}\xi^{(s)}$$

$$-\sum_{s=1}^{k+j}\phi^{(s)} \sum_{m=\left\{\begin{smallmatrix}0\\ j-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}k+j-s\\ j-1\end{smallmatrix}\right.} \frac{j\,!\;k\,!\;a_m^{(k+j-s-m)}}{(j-m)\,!\;m\,!\;(k+j-s-m)\,!\;(s+m-j)\,!}$$

$$+\sum_{s=2}^{k+j+1}\xi^{(s)} \sum_{m=\left\{\begin{smallmatrix}0\\ j+1-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j+1-s\end{smallmatrix}\right.} \frac{j\,!\;k\,!\;(n-j)\,a_m^{(k+j+1-s-m)}}{(j+1-m)\,!\;m\,!\;(k+j+1-s-m)\,!\;(m+s-j-1)\,!}\,. \quad (27)$$

$$(j=1,\,2,\,\ldots,\,n;\; k=0,\,1,\,2,\,\ldots;\; a_0=1)$$

Formula (27) includes (25) as a special case, so that we need use only (27) and (24) in forming the extended group.

§25. It has been already stated that $a_j^{(k)}$ has the weight $(k+j)$, and that $y^{(\mu)}$ has the weight μ. If then we consider only those extended transformations which involve no $y^{(\mu)}$ of weight greater than w_1 and no $a_j^{(k)}$ of weight greater than w_2, we have as the required extended transformation

$$X^{(w_1 w_2)}f = Xf + \sum_{k=1}^{w_1}\frac{\delta y^{(k)}}{\delta t}\cdot\frac{\partial f}{\partial y^{(k)}} + \sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n\end{smallmatrix}\right.}\sum_{k=0}^{w_2-j}\frac{\delta a_j^{(k)}}{\delta t}\cdot\frac{\partial f}{\partial a_j^{(k)}}\,.$$

In this formula the values of $\frac{\delta y^{(k)}}{\delta t}$ from (24) and of $\frac{\delta a_j^{(k)}}{\delta t}$ from (27) are to be substituted. The result will involve double, triple and quadruple summations. The order of summation must then be so changed that the summations with regard to $\xi^{(s)}$ and $\phi^{(s)}$ come first. When this is done, we get *the extended transformation*:

$$X^{(w_1 w_2)}f = \xi\frac{\partial f}{\partial x} + \sum_{s=0}^{w_1}\phi^{(s)}\sum_{k=s}^{w_1}\frac{k\,!\;y^{(k-s)}}{(k-s)\,!\;s\,!}\,\frac{\partial f}{\partial y^{(k)}} - \sum_{s=1}^{w_1}\xi^{(s)}\sum_{k=s}^{w_1}\frac{k\,!\;y^{(k-s+1)}}{(k-s)\,!\;s\,!}\cdot\frac{\partial f}{\partial y^{(k)}}$$

$$-\sum_{s=1}^{w_2}\phi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n\end{smallmatrix}\right.}\sum_{k=\left\{\begin{smallmatrix}s-j\\ 0\end{smallmatrix}\right.}^{w_2-j}\sum_{m=\left\{\begin{smallmatrix}0\\ j-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j-s\end{smallmatrix}\right.}\frac{j!\;k!\;a_m^{(k+j-s-m)}}{(j-m)!\;m!\;(k+j-s-m)!\;(s+m-j)!}\,\frac{\partial f}{\partial a_j^{(k)}}$$

$$-\sum_{s=1}^{w_2}\xi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}n\\ w_2-s+1\end{smallmatrix}\right.}\sum_{k=s-1}^{w_2-j}\frac{k!\;[k+1+s(j-1)]}{(k+1-s)!\;s!}\,a_j^{(k+1-s)}\,\frac{\partial f}{\partial a_j^{(k)}}$$

$$+\sum_{s=2}^{w_2+1}\xi^{(s)}\sum_{j=1}^{\left\{\begin{smallmatrix}w_2\\ n-1\end{smallmatrix}\right.}\sum_{k=\left\{\begin{smallmatrix}0\\ s-j-1\end{smallmatrix}\right.}^{w_2-j}\sum_{m=\left\{\begin{smallmatrix}j+1-s\\ 0\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix}j-1\\ k+j+1-s\end{smallmatrix}\right.}$$

$$\frac{j!\;k!\;(n-j)\,a_m^{(k+j+1-s-m)}}{(j+1-m)!\;m!\;(k+j+1-s-m)!\;(m+s-j-1)!}\,\frac{\partial f}{\partial a_j^{(k)}}\,. \quad (28)$$

§26. If $F(y^{(\mu)}, a_j^{(k)})$ is to be invariant for this transformation, then $X^{(w_1 w_2)}F = 0$. This equation breaks up into a number of equations, obtained by equating the coefficients of $\phi^{(s)}$ and $\xi^{(s)}$ to zero, for all possible values of s. The general case will not be here considered, but only the two special cases, $w_1 = 0$, $w_2 = w$, and $w_1 = w_2 = w$. The first of these will yield what are ordinarily known as invariants, that is, the invariant functions so obtained will not involve y or its derivatives; the second yields, in addition to those of the first class, the so-called covariants. In writing these equations those coming from $\phi^{(s)}$ are always given first, and then those from $\xi^{(s)}$. The former class of equations yield the invariants for transformations of y alone, the latter the invariants for transformations of x alone, while the common solutions of both sets of equations are the invariants for change of both x and y. Either set of equations alone forms a *complete* system. We have then the following complete system of

Equations to be satisfied by invariants involviny only $a_j^{(k)}$, where $k + j \leqq w$, for transformation of both x and y:

$$\left.\begin{aligned}
0 = -B_s f \equiv{}& \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n \end{smallmatrix}\right.} \sum_{k=\left\{\begin{smallmatrix} s-j \\ 0 \end{smallmatrix}\right.}^{w-j} \sum_{m=\left\{\begin{smallmatrix} 0 \\ j-s \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.} \frac{j!\,k!\,a_m^{(k+j-s-m)}}{(j-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
& (s = 1, 2, \ldots, w) \\
0 = -A_1 f \equiv{}& \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n \end{smallmatrix}\right.} \sum_{k=0}^{w-j} (k+j)\,a_j^{(k)}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
0 = -A_{s+1} f \equiv{}& \sum_{j=1}^{\left\{\begin{smallmatrix} w-s \\ n \end{smallmatrix}\right.} \sum_{k=s}^{w-j} \frac{k!\,[k+j+s(j-1)]}{(k-s)!\,(s+1)!}\,a_j^{(k-s)}\,\frac{\partial f}{\partial a_j^{(k)}} \\
& - \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n-1 \end{smallmatrix}\right.} \sum_{k=\left\{\begin{smallmatrix} 0 \\ s-j \end{smallmatrix}\right.}^{w-j} \sum_{m=\left\{\begin{smallmatrix} j-s \\ 0 \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.} \frac{j!\,k!\,(n-j)\,a_m^{(k+j-s-m)}}{(j+1-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!}\,\frac{\partial f}{\partial a_j^{(k)}}, \\
& (s = 1, 2, \ldots, w-1) \\
0 = A_{w+1} f \equiv{}& \sum_{j=1}^{\left\{\begin{smallmatrix} w \\ n-1 \end{smallmatrix}\right.} \frac{n-j}{j+1}\,\frac{\partial f}{\partial a_j^{(w-j)}}.
\end{aligned}\right\} (29)$$

In using these equations it is to be remembered that $a_0 = 1$, so that $a_0^{(k)} = 0$ when $k > 0$.

In the same manner we get the complete system of

Equations to be satisfied by invariants involving $a_j^{(k)}$ $(k+j\leqq w)$ *and* $y^{(\mu)}$ $(\mu\leqq w)$ *for transformation of both variables* x *and* y.

$$\left.\begin{aligned}
0 &= Y_0 f \equiv \sum_{k=0}^{w} y^{(k)} \frac{\partial f}{\partial y^{(k)}}, \\
0 &= Y_s f \equiv \sum_{k=s}^{w} \frac{k!}{(k-s)!\;s!}\, y^{(k-s)} \frac{\partial f}{\partial y^{(k)}} \\
&\quad -\sum_{j=1}^{\left\{\substack{w\\n}\right.} \sum_{k=\left\{\substack{s-j\\0}\right.}^{w-j} \sum_{m=\left\{\substack{0\\j-s}\right.}^{\left\{\substack{j-1\\k+j-1}\right.} \frac{j!\;k!\;a_m^{(k+j-s-m)}}{(j-m)!\;m!\;(k+j-s-m)!\;(m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad\qquad (s=1,\,2,\,\ldots\,.\,,\,w) \\
0 &= -X_1 f \equiv \sum_{k=1}^{w} k y^{(k)} \frac{\partial f}{\partial y^{(k)}} + \sum_{j=1}^{\left\{\substack{w\\n}\right.} \sum_{k=0}^{w-j} (k+j)\, a_j^{(k)} \frac{\partial f}{\partial a_j^{(k)}}, \\
0 &= -X_{s+1} f \equiv \sum_{k=s+1}^{w} \frac{k!\;y^{(k-s)}}{(k-s-1)!\;(s+1)!} \frac{\partial f}{\partial y^{(k)}} \\
&\quad + \sum_{j=1}^{\left\{\substack{w-s\\n}\right.} \sum_{k=s}^{w-j} \frac{k!\;[k+j+s(j-1)]}{(k-s)!\;(s+1)!}\, a_j^{(k-s)} \frac{\partial f}{\partial a_j^{(k)}} \\
&\quad - \sum_{j=1}^{\left\{\substack{w\\n-1}\right.} \sum_{k=\left\{\substack{0\\s-j}\right.}^{w-j} \sum_{m=\left\{\substack{j-s\\0}\right.}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\;k!\;(n-j)\,a_m^{(k+j-s-m)}}{(j+1-m)!\;m!\;(k+j-s-m)!\;(m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad\qquad (s=1,\,2,\,\ldots\,.\,,\,w-1) \\
0 &= X_{w+1} f \equiv \sum_{j=1}^{\left\{\substack{w\\n-1}\right.} \frac{n-j}{j+1} \frac{\partial f}{\partial a_j^{(w-j)}}\,.
\end{aligned}\right\} (30)$$

§27. Using the notation of (30) we may write (28) in the form

$$X^{(ww} f = \xi \frac{\partial f}{\partial x} + \sum_{s=0}^{w} \phi^{(s)} Y_s f + \sum_{s=1}^{w+1} \xi^{(s)} X_s f. \tag{31}$$

The equations $Y_s f = 0$ in (30) are, of course, the same as the equations (9) which yield the covariants for transformation of y alone. The equations $X_s f = 0$ in (30) yield the covariants for transformation of x alone.

§28. The equation $0 = -A_1 f$ of the system (29) shows that every solution of this system must be an isobaric function of the $a_j^{(k)}$'s of weight zero. Every rational invariant must therefore be the quotient of two integral rational functions, which are isobaric of the same weight. The equation $0 = - X_1 f$ of (30) shows that the covariants must also be isobaric of weight zero in the $a_j^{(k)}$'s and $y^{(\mu)}$'s, and the equation $Y_0 f = 0$ that they must be homogeneous of the zeroth degree in $y, y', \ldots\ldots, y^{(w)}$. (Cf. Theorems I and III.)

§29. The complete system of partial differential equations (29) contains $2w + 1$ equations. When $w \leqq n$ there are $\frac{1}{2} w(w+1)$ variables. Hence there are at least

$$\tfrac{1}{2} w(w+1) - (2w+1) = \tfrac{1}{2}(w^2 - 3w - 2)$$

solutions. In order to have any solutions for this case we must have $n \geqq w \geqq 4$. When $w > n$ there are $nw - \frac{1}{2} n(n-1)$ variables, and therefore

$$nw - \tfrac{1}{2} n(n-1) - (2w+1) = (n-2)w - \tfrac{1}{2}(n^2 - n - 2)$$

solutions. When $n = 2$ this is always negative, however great w be chosen, which shows, if we assume the equations to be independent, that the linear differential equation of the second order has no invariants, a fact which has long been known. When $n = 3$ there are $w - 4$ solutions, so that there is certainly a solution for $w = 5$, that is, involving terms of weight five. We shall return to this later. Similarly we see that there is one solution involving only a_1, a_2, a_3 and their derivatives to those of weight 5, whatever n may be. Call this invariant I_3. Either its numerator or denominator involve terms of weight 5. Then making $w = 4$ in the formula $\frac{1}{2}(w^2 - 3w - 2)$ for the number of solutions, we see that there is one solution which must involve a_4. Call this solution I_4. When $w = 5$ there are four solutions, one of which involves a_5, and which we shall call I_5. And in general call that solution I_m which involves a_m and a's of smaller subscript, with their derivatives up to those of weight m. When $w \leqq m$ we can in this manner pick out a set of $w - 2$ independent solutions, from which we may obtain the rest by differentiation. These solutions which we have just chosen will be algebraic, and in all of the examples which follow they may be made rational by simply raising to a power. If we assume then that

$$I_m = \frac{U^{(m)}}{V^{(m)}},$$

(21) shows that

$$\Delta_m I_m = \left[\frac{1}{V^{(m)}}\right]^{\frac{1}{m}} \frac{d}{dx} I_m$$

is an invariant, and

$$I_m, \quad \Delta_m I_m, \quad \Delta_m^2 I_m, \ldots\ldots, \Delta_m^{w-m} I_m; \qquad (m = 4, 5, \ldots\ldots, w)$$

is a set of $w - m + 1$ invariants, all independent, and none involving terms of weight greater than w. The invariant I_3 is not included in this set, and can be differentiated only $w - 5$ times, since it involves terms of weight 5, not 3. It therefore furnishes only $w - 4$ solutions. By subjecting

$$I_3\,;\; I_4,\, I_5, \ldots\ldots, I_w$$

to this process we thus get

$$w - 4 + [(w-3) + (w-4) + (w-5) + \ldots\ldots + 2 + 1] = \tfrac{1}{2}(w^2 - 3w - 2)$$

invariants which are evidently independent; that is, this process gives all the invariants. The system of $w - 2$ invariants

$$I_3,\, I_4, \ldots\ldots, I_w$$

is then a *complete system of invariants* from which all others may be derived by differentiation.

When $w > n$ the same process applied to

$$I_3,\, I_4, \ldots\ldots, I_n$$

yields

$$w - 4 + [(w-3) + (w-4) + \ldots\ldots + (w-n+1)] = (n-2)\,w - \tfrac{1}{2}(n^2 - n + 2)$$

invariants, and we have here all the solutions.

§30. The complete system of partial differential equations (30) contains $2w + 2$ equations. When $w < n$ the number of variables is $w + 1 + \frac{1}{2}w\,(w+1)$, and hence there are at least $\frac{1}{2}(w-2)(w+1)$ independent solutions. Of these solutions $\frac{1}{2}(w^2 - 3w - 2)$ are solutions of (29), so that we have still w to find. An examination of (30) shows that there is one solution involving y and y' and a's of weight not greater than 4. The differentiation process may be applied $w - 4$ times to this covariant, which gives us all but the three covariants of highest order. The process cannot be applied oftener because it would then bring in a's of weight greater than w. We shall presently give the solution of (30) for $w = 3$. This gives two covariants involving y, y', y'', y''' and a's of weight not greater than 3. The application of the differentiation process to these two yields the whole series of covariants except the one containing only y and y', which must be determined by a separate integration. This latter covariant is also given later.

§31. When, in (30), $w \leqq n$, there are $nw - \frac{1}{2}n(n-1) + w + 1$ variables, but they are not all independent, being bound by the given differential equation (2), so that $y^{(n)}$ may always be eliminated from a covariant of order in y equal to or greater than n. Moreover, the covariants of order in y higher than n have little interest, for they can always be found by a differentiation process. The given equation (2) is an invariant equation, and, as will be shown later.

$$2a_3 - 3a_2' - 6a_1a_2 + 4a_1^3 + 6a_1a_1' + a_1''$$

is also a relative invariant. Hence by §17

$$\frac{y^{(n)} + na_1y^{(n-1)} + \dfrac{n\,(n-1)}{2}\,a_2\,y^{(n-2)} + \ldots\ldots + a_ny}{y\left[2a_3 - 3a_2' - 6a_1a_2 + 4a_1^3 + 6a_1a_1' + a_1''\right]^{\frac{n}{3}}}$$

is an absolute covariant (equal to zero when we use (2)). The repeated application of (2) to this function will give *all* the covariants of order in y greater than n. The quantity $y^{(n)}$ must then be eliminated from these by means of (2). We shall leave these covariants out of consideration, and when $w_2 \leqq n$, take $w_1 = n - 1$. These values in (28) show that these covariants are the solutions of the following complete system:

$$\left.\begin{aligned}
0 = \quad Y_0 f \quad &\equiv \sum_{k=0}^{n-1} y^{(k)} \frac{\partial f}{\partial y^{(k)}}, \\
0 = \quad Y_s f \quad &\equiv \sum_{k=s}^{n-1} \frac{k!}{(k-s)!\, s!} y^{(k-s)} \frac{\partial f}{\partial y^{(k)}} \\
&\quad - \sum_{j=1}^{n} \sum_{k=\left\{\substack{s-j\\0}\right.}^{w-j} \sum_{m=\left\{\substack{0\\j-s}\right.}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\; k!\; a_m^{(k+j-s-m)}}{(j-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1,\ 2,\ \ldots\ldots,\ n-1) \\
0 = - Y_s f \quad &\equiv \sum_{j=1}^{n} \sum_{k=s-j}^{w-j} \sum_{m=0}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\; k!\; a_m^{(k+j-s-m)}}{(j-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = n,\ n+1,\ \ldots\ldots,\ w) \\
0 = - X_1 f \quad &\equiv \sum_{k=1}^{n-1} k y^{(k)} \frac{\partial f}{\partial y^{(k)}} + \sum_{j=1}^{n} \sum_{k=0}^{w-j} (k+j) a_j^{(k)} \frac{\partial f}{\partial a_j^{(k)}}, \\
0 = - X_{s+1} f &\equiv \sum_{k=s+1}^{n-1} \frac{k!\; y^{(k-s)}}{(k-s-1)!\; (s+1)!} \frac{\partial f}{\partial y^{(k)}} \\
&\quad + \sum_{j=1}^{\left\{\substack{n\\w-s}\right.} \sum_{k=s}^{w-j} \frac{k!\; [k+j+s(j-1)]}{(k-s)!\; (s+1)!} a_j^{(k-s)} \frac{\partial f}{d a_j^{(k)}} \\
&\quad - \sum_{j=1}^{n-1} \sum_{k=\left\{\substack{0\\s-j}\right.}^{w-j} \sum_{m=\left\{\substack{j-s\\0}\right.}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\; k!\; (n-j)\, a_m^{(k+j-s-m)}}{(j+1-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = 1,\ 2,\ \ldots\ldots,\ n-2) \\
0 = - X_{s+1} f &\equiv \sum_{j=1}^{\left\{\substack{n\\w-s}\right.} \sum_{k=s}^{w-j} \frac{k!\; [k+j+s(j-1)]}{(k-s)!\; (s+1)!} a_j^{(k-s)} \frac{\partial f}{\partial a_j^{(k)}} \\
&\quad - \sum_{j=1}^{n-1} \sum_{k=s-j}^{w-j} \sum_{m=0}^{\left\{\substack{j-1\\k+j-s}\right.} \frac{j!\; k!\; (n-j)\, a_m^{(k+j-s-m)}}{(j+1-m)!\; m!\; (k+j-s-m)!\; (m+s-j)!} \frac{\partial f}{\partial a_j^{(k)}}, \\
&\qquad (s = n-1,\ n,\ \ldots\ldots,\ w-1) \\
0 = X_{w+1} f &\equiv \sum_{j=1}^{n-1} \frac{n-j}{j+1} \frac{\partial f}{\partial a_j^{(w-j)}}.
\end{aligned}\right\} (32)$$

The number of solutions of these equations is

$$nw - \tfrac{1}{2}n(n-3) - (2w+2) = (n-2)w - \tfrac{1}{2}(n^2 - 3n + 4).$$

Of these all but $n-1$ are solutions of (29). These $n-1$ are found just as for the case $w < n$, and in fact are the same solutions as for $w = n-1$.

§32. As an example, let us give the solutions of (30) for $w = 3$. The equations become:

$$0 = \quad Y_0 f \equiv y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y''\frac{\partial f}{\partial y''} + y'''\frac{\partial f}{\partial y'''},$$

$$0 = \quad Y_1 f \equiv y\frac{\partial f}{\partial y'} + 2y'\frac{\partial f}{\partial y''} + 3y''\frac{\partial f}{\partial y'''} - \frac{\partial f}{\partial a_1} - 2a_1\frac{\partial f}{\partial a_2} - 2a_1'\frac{\partial f}{\partial a_2'} - 3a_2\frac{\partial f}{\partial a_3},$$

$$0 = \quad Y_2 f \equiv y\frac{\partial f}{\partial y''} + 3y'\frac{\partial f}{\partial y'''} - \frac{\partial f}{\partial a_1'} - \frac{\partial f}{\partial a_2} - 2a_1\frac{\partial f}{\partial a_2'} - 3a_1\frac{\partial f}{\partial a_3},$$

$$0 = \quad Y_3 f \equiv y\frac{\partial f}{\partial y'''} - \frac{\partial f}{\partial a_1''} - \frac{\partial f}{\partial a_2'} - \frac{\partial f}{\partial a_3},$$

$$0 = -X_1 f \equiv y'\frac{\partial f}{\partial y'} + 2y''\frac{\partial f}{\partial y''} + 3y'''\frac{\partial f}{\partial y'''} + a_1\frac{\partial f}{\partial a_1} + 2a_1'\frac{\partial f}{\partial a_1'} + 3a_1''\frac{\partial f}{\partial a_1''} + 2a_2\frac{\partial f}{\partial a_2} + 3a_2'\frac{\partial f}{\partial a_2'} + 3a_3\frac{\partial f}{\partial a_3},$$

$$0 = -X_2 f \equiv y'\frac{\partial f}{\partial y''} + 3y''\frac{\partial f}{\partial y'''} - \frac{n-1}{2}\frac{\partial f}{\partial a_1} + a_1\frac{\partial f}{\partial a_1'} + 3a_1'\frac{\partial f}{\partial a_1''} - (n-2)a_1\frac{\partial f}{\partial a_2} + [2a_2 - (n-2)a_1']\frac{\partial f}{\partial a_2'} - \frac{3(n-3)}{2}a_2\frac{\partial f}{\partial a_3},$$

$$0 = -X_3 f \equiv y'\frac{\partial f}{\partial y'''} - \frac{n-1}{2}\frac{\partial f}{\partial a_1'} + a_1\frac{\partial f}{\partial a_1''} - \frac{n-2}{3}\frac{\partial f}{\partial a_2} - (n-2)a_1\frac{\partial f}{\partial a_2'} - (n-3)a_1\frac{\partial f}{\partial a_3},$$

$$0 = \quad X_4 f \equiv \frac{n-1}{2}\frac{\partial f}{\partial a_1''} + \frac{n-2}{3}\frac{\partial f}{\partial a_2'} + \frac{n-3}{4}\frac{\partial f}{\partial a_3}.$$

This complete system of 8 equations (which are independent) in 10 variables has two solutions. By the ordinary methods they are found to be

$$\frac{(n-1)yy'' - (n-2)y'^2 + 2a_1yy' + \dfrac{(n-1)^2}{n+1}\left[3a_2 - \dfrac{2(n-2)}{n-1}a_1' - \dfrac{(n-2)(3n-1)}{(n-1)^2}a_1^2\right]y^2}{y^2(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'')^{\frac{2}{3}}},$$

and

$$\frac{\left[y''' + \frac{6}{n-1} a_1 y'' + \frac{12}{n+1} a_2 y' + \frac{2(n-1)}{n+1} a_3 y\right] y^2 + (n-3) \left\{ \begin{array}{l} \frac{-3}{n-1} y'' y' y + \frac{2(n-2)}{(n-1)^2} y'^3 - \frac{6}{(n-1)^2} a_1 y'^2 y \\ \qquad + \frac{6}{n^2-1} a_1' y' y^2 + \frac{12n}{(n-1)^2(n+1)} a_1^2 y' y^2 \\ + \left[4 \frac{n^2-3n+1}{(n-1)^2(n+1)} a_1^3 - \frac{6}{n+1} a_2 a_1 - \frac{6}{n^2-1} a_1 a_1' - \frac{1}{n+1} a_1''\right] y^3 \end{array} \right\}}{y^3 (2a_3 - 3a_2' - 6a_2 a_1 + 4a_1^3 + 6a_1 a_1' + a_1'')}.$$

Both numerator and denominator of both of these invariants are relative invariants (Th. III), and this justifies the use of the denominator made in §31. When $n = 3$, the numerator of the second invariant reduces to

$$(y''' + 3a_1 y'' + 3a_2 y' + a_3 y) y^2,$$

as it should.

These are the two covariants mentioned in §30 from which all others (except the one involving only y and y') may be derived by differentiation.

The covariant involving only y and y' is the solution of the following complete system, which is obtained by writing (30) for $w = 4$ and then dropping the derivatives with regard to y'', y''', y^{IV} and a_4.

$$0 = y \frac{\partial f}{\partial y} + y' \frac{\partial f}{\partial y'},$$

$$0 = y \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial a_1} - 2a_1 \frac{\partial f}{\partial a_2} - 2a_1' \frac{\partial f}{\partial a_2'} - 2a_1'' \frac{\partial f}{\partial a_2''} - 3a_2 \frac{\partial f}{\partial a_3} - 3a_2' \frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1'} + \frac{\partial f}{\partial a_2} + 2a_1 \frac{\partial f}{\partial a_2'} + 4a_1' \frac{\partial f}{\partial a_2''} + 3a_1 \frac{\partial f}{\partial a_3} + 3(a_2 + a_1') \frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1''} + \frac{\partial f}{\partial a_2'} + \frac{\partial f}{\partial a_3} + 3a_1 \frac{\partial f}{\partial a_3'},$$

$$0 = \frac{\partial f}{\partial a_1'''} + \frac{\partial f}{\partial a_2''} + \frac{\partial f}{\partial a_3'},$$

$$0 = y' \frac{\partial f}{\partial y'} + a_1 \frac{\partial f}{\partial a_1} + 2a_1' \frac{\partial f}{\partial a_1'} + 3a_1'' \frac{\partial f}{\partial a_1''} + 4a_1''' \frac{\partial f}{\partial a_1'''} + 2a_2 \frac{\partial f}{\partial a_2} + 3a_2' \frac{\partial f}{\partial a_2'}$$
$$+ 4a_2'' \frac{\partial f}{\partial a_2''} + 3a_3 \frac{\partial f}{\partial a_3} + 4a_3' \frac{\partial f}{\partial a_3'},$$

$$
\begin{aligned}
0 = &-\frac{n-1}{2}\frac{\partial f}{\partial a_1} + a_1\frac{\partial f}{\partial a_1'} + 3a_1'\frac{\partial f}{\partial a_1''} + 6a_1''\frac{\partial f}{\partial a_1'''} - (n-2)\,a_1\frac{\partial f}{\partial a_2} \\
&+ [2a_2 - (n-2)\,a_1']\frac{\partial f}{\partial a_2'} + [5a_2' - (n-2)\,a_1'']\frac{\partial f}{\partial a_2''} \\
&- \frac{3\,(n-3)}{2}\,a_2\frac{\partial f}{\partial a_3} + \left[3a_3 - \frac{3\,(n-3)}{2}\,a_2'\right]\frac{\partial f}{\partial a_3'},
\end{aligned}
$$

$$
\begin{aligned}
0 = &-\frac{n-1}{2}\frac{\partial f}{\partial a_1'} + a_1\frac{\partial f}{\partial a_1''} + 4a_1'\frac{\partial f}{\partial a_1'''} - \frac{n-2}{3}\frac{\partial f}{\partial a_2} - (n-2)\,a_1\frac{\partial f}{\partial a_2'} \\
&+ 2\,[a_2 - (n-2)\,a_1']\frac{\partial f}{\partial a_2''} - (n-3)\,a_1\frac{\partial f}{\partial a_3} - (n-3)(\tfrac{3}{2}\,a_2 + a_1')\frac{\partial f}{\partial a_3'},
\end{aligned}
$$

$$
\begin{aligned}
0 = &-\frac{n-1}{2}\frac{\partial f}{\partial a_1''} + a_1\frac{\partial f}{\partial a_1'''} - \frac{n-2}{3}\frac{\partial f}{\partial a_2'} - (n-2)\,a_1\frac{\partial f}{\partial a_2''} \\
&- \frac{n-3}{4}\frac{\partial f}{\partial a_3} - (n-3)\,a_1\frac{\partial f}{\partial a_3'},
\end{aligned}
$$

$$
0 = \frac{n-1}{2}\frac{\partial f}{\partial a_1'''} + \frac{n-2}{3}\frac{\partial f}{\partial a_2''} + \frac{n-3}{4}\frac{\partial f}{\partial a_3'}.
$$

The one solution of this system of equations is

$$
\frac{\begin{aligned}6\,(y' + a_1y)(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'') \\ + (n-1)\,y\,(2a_3' - 3a_2'' - 6a_2'a_1 - 6a_2a_1' + 12a_1^2a_1' + 6a_1a_1'' + 6a_1'^2 + a_1''')\end{aligned}}{y\,(2a_3 - 3a_2' - 6a_2a_1 + 4a_1^3 + 6a_1a_1' + a_1'')^{\frac{4}{3}}}.
$$

CHAPTER 4.

Consideration of a Subgroup.

§33. Let us now turn to the consideration of a case which within the last few years has received the attention of a number of writers, and which was perhaps first completely treated by Forsyth.* If in the equation

$$
y^{(n)} + na_1y^{(n-1)} + \ldots. + a_ny = 0 \tag{2}
$$

we make the transformation

$$
y = ze^{-\int a_1\,dx},
$$

---

* Philosophical Transactions, 1888, I, "On Invariants, etc., associated with the Linear Differential Equation," pp. 377-489.

the term involving $z^{(n-1)}$ disappears, and we get

$$z^{(n)} + \frac{n(n-1)}{2} p_2 z^{(n-2)} + \dots + p_n z = 0, \tag{33}$$

where $p_2, p_3, \dots p_n$ are functions of x alone. If in this equation x be subjected to an arbitrary transformation

$$x_1 = \chi(x),$$

we can afterwards cause the term involving the $(n-1)^{\text{th}}$ derivative to disappear by a transformation of the form

$$z_1 = z\Psi(x).$$

All such transformations must form a subgroup of the group (11), for they are included in it and leave the form of (33) unaltered. We may then seek the invariants of (33) for the group of transformations of this form. The infinitesimal transformations of this subgroup are readily found. From (27) we have

$$\delta a_1 = -\delta t\left[\xi' a_1 + \phi' - \frac{n-1}{2}\xi''\right].$$

In the case under consideration $a_1 = 0$ and $\delta a_1 = 0$. Hence we must have the relation

$$\phi' = \frac{n-1}{2}\xi''. \tag{34}$$

This may be taken as the defining equation of the subgroup in question. Integrated, it is

$$\phi = \frac{n-1}{2}\xi' + C, \qquad (C = \text{constant})$$

§34. When the preceding value of ϕ is substituted in (24), the result is

$$\delta z^{(k)} = \left[Cz^{(k)} + \tfrac{1}{2}\sum_{0}^{k}{}_{m} \frac{k!\,[(m+1)(n+1) - 2(k+1)]}{(k-m)!\,(m+1)!}\xi^{(m+1)} z^{(k-m)}\right]\delta t. \tag{35}$$

Similarly (26) becomes

$$\frac{\delta p_j^{(k)}}{\delta t} = -\sum_{0}^{k}{}_{s} \frac{k!\,[j(k+1) - s(j-1)]}{(k-s+1)!\,s!} p_j^{(s)} \xi^{(k-s+1)}$$
$$-\tfrac{1}{2}\sum_{0}^{j-1}{}_{s} \frac{j!\,[n(j-s-1) + j + s - 1]}{(j-s+1)!\,s!} \frac{d^k}{dx^k}(\xi^{(j-s+1)} p_s). \tag{36}$$

$(j = 2, 3, \dots, n;\ k = 0, 1, 2, \dots;\ p_0 = 1,\ p_1 = 0)$

This is Forsyth's (13). Our (27) becomes

$$\frac{\delta p_j^{(k)}}{\delta t} = -\mathop{\sum\nolimits_s}\limits_{1}^{k+1} \frac{k!\ [k+1+s(j-1]}{(k-s+1)!\ s!}\, p_j^{(k-s+1)}\xi^{(s)}$$

$$-\tfrac{1}{2}\mathop{\sum\nolimits_s}\limits_{2}^{k+j+1}\xi^{(s)}\mathop{\sum\nolimits_m}\limits_{\left\{\begin{smallmatrix} j+1-s \\ 0\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j+1-s\end{smallmatrix}\right.} \frac{j!\,k!\,[n(j-m-1)+j+m-1]\,p_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}. \quad (37)$$

$$(j = 2, 3, \ldots\ldots, n;\ k = 0, 1, 2, \ldots\ ;\ p_0 = 1,\ p_1 = 0)$$

For $j = 1$, (37) gives

$$\frac{\delta p_1^{(k)}}{\delta t} = -\mathop{\sum\nolimits_s}\limits_{1}^{k+1} \frac{(k+1)!}{(k-s+1)!\ s!}\, p_1^{(k-s+1)}\xi^{(s)} \equiv 0, \text{ since } p_1 \equiv 0.$$

§35. The extended transformation, corresponding to (28), is

$$Z^{(w_1 w_2)}f = \xi\frac{\partial f}{\partial x} + C\mathop{\sum\nolimits_k}\limits_{0}^{w_1} z^{(k)}\frac{\partial f}{\partial z^{(k)}} + \tfrac{1}{2}\mathop{\sum\nolimits_s}\limits_{1}^{w_1+1}\xi^{(s)}\mathop{\sum\nolimits_k}\limits_{s-1}^{w_1}\frac{k!\ [s(n+1)-2(k+1)]}{s!\ (k-s+1)!}\, z^{(k-s+1)}\frac{\partial f}{\partial z^{(k)}}$$

$$-\mathop{\sum\nolimits_s}\limits_{1}^{w_2-1}\xi^{(s)}\mathop{\sum\nolimits_j}\limits_{2}^{\left\{\begin{smallmatrix} n \\ w_2-s+1\end{smallmatrix}\right.}\mathop{\sum\nolimits_k}\limits_{s-1}^{w_2-j}\frac{k!\ [k+1+s(j-1)]}{(k-s+1)!\ s!}\, p_j^{(k-s+1)}\frac{\partial f}{\partial p_j^{(k)}} \quad (38)$$

$$-\tfrac{1}{2}\mathop{\sum\nolimits_s}\limits_{2}^{w_2+1}\xi^{(s)}\mathop{\sum\nolimits_j}\limits_{2}^{\left\{\begin{smallmatrix} n \\ w_2\end{smallmatrix}\right.}\mathop{\sum\nolimits_k}\limits_{\left\{\begin{smallmatrix} s-j-1 \\ 0\end{smallmatrix}\right.}^{w_2-j}\mathop{\sum\nolimits_m}\limits_{\left\{\begin{smallmatrix} 0 \\ j+1-s\end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j+1-s\end{smallmatrix}\right.}\frac{j!\,k!\,[n(j-m-1)+j+m-1]\,p_m^{(k+j+1-s-m)}}{(j+1-m)!\,m!\,(k+j+1-s-m)!\,(m+s-j-1)!}\frac{\partial f}{\partial p_j^{(k)}},$$

To get the equations whose solutions are the invariants we must equate the coefficients of $\xi^{(s)}$, and also that of C, to zero. As before, the two cases $w_1 = 0$, $w_2 = w$ and $w_1 = w_2 = w$ are considered. We find thus the following complete systems of equations:

Equations to be satisfied by invariants involving only $p_j^{(k)}$ $(k+j \leqq w)$, *for the subgroup*, $\phi' = \frac{n-1}{2}\xi''$.

$$\left.\begin{aligned}
0 &= -A_1'f \equiv \sum_{j=2}^{\left\{\begin{smallmatrix} w \\ n \end{smallmatrix}\right.} \sum_{k=0}^{w-j} (k+j)\, p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}}, \\
0 &= -2A_{s+1}'f \equiv 2 \sum_{j=2}^{\left\{\begin{smallmatrix} n \\ w-s \end{smallmatrix}\right.} \sum_{k=s}^{w-j} \frac{k!\,[k+j+s(j-1)]}{(k-s)!\,(s+1)!}\, p_j^{(k-s)} \frac{\partial f}{\partial p_j^{(k)}} \\
&\quad + \sum_{j=2}^{\left\{\begin{smallmatrix} n \\ w \end{smallmatrix}\right.} \sum_{k=\left\{\begin{smallmatrix} 0 \\ s-j \end{smallmatrix}\right.}^{w-j} \sum_{m=\left\{\begin{smallmatrix} j-s \\ 0 \end{smallmatrix}\right.}^{\left\{\begin{smallmatrix} j-1 \\ k+j-s \end{smallmatrix}\right.} \frac{j!\,k!\,[n(j-m-1)+j+m-1]\, p_m^{(k+j-s-m)}}{(j+1-m)!\,m!\,(k+j-s-m)!\,(m+s-j)!} \frac{\partial f}{\partial p_j^{(k)}}, \\
&\qquad (s = 1, 2, \ldots, w-2) \\
0 &= -\frac{2}{n+1} A_{w+1-s}'f \equiv \sum_{j=2}^{\left\{\begin{smallmatrix} n \\ w-s \end{smallmatrix}\right.} \frac{j-1}{j+1} \frac{\partial f}{\partial p_j^{(w-j-s)}}. \qquad (s = 0, 1)
\end{aligned}\right\} (39)$$

Equations to be satisfied by invariants involving $p_j^{(k)}$, $(k+j \leqq w)$ *and* $z^{(l)}$ $(l \leqq w)$, *for the subgroup* $\phi' = \frac{n-1}{2}\xi''$.

$$\left.\begin{aligned}
0 &= \quad Y_0'f \equiv \sum_{k=0}^{w} z^{(k)} \frac{\partial f}{\partial z^{(k)}}, \\
0 &= -X_1'f + \frac{n-1}{2} Y_0'f \equiv \sum_{k=1}^{w} k z^{(k)} \frac{\partial f}{\partial z^{(k)}} - A_1'f, \\
0 &= \quad 2X_{s+1}'f \equiv \sum_{k=s}^{w} \frac{k!\,[s(n+1)+n-2k-1]}{(s+1)!\,(k-s)!}\, z^{(k-s)} \frac{\partial f}{\partial z^{(k)}} + 2A_{s+1}'f, \\
&\qquad (s = 1, 2, \ldots, w)
\end{aligned}\right\} (40)$$

In making use of (40) the value of $A_{s+1}'f$ must be substituted from (39), and in using either (39) or (40) we must remember that $p_0 = 1$, $p_1 = 0$.

§36. The complete system of partial differential equations (39) contains $w+1$ equations, and, when $w \leqq n$, $\frac{1}{2}w(w-1)$ variables. It has therefore at least $\frac{1}{2}(w^2-3w-2)$ solutions. These are of course the $\frac{1}{2}(w^2-3w-2)$ solutions of (29) for $a_1^{(k)} = 0$, $a_j^{(k)} = p_j^{(k)}$ $(j > 1)$. Similarly when $w > n$ there are $(n-2)w - \frac{1}{2}(n^2-n+2)$ solutions, which are obtained from the $(n-2)w$

$-\frac{1}{2}(n^2-n+2)$ solutions of (29) for $w>n$ by writing $a_1^{(k)}=0$, $a_j^{(k)}=p_j^{(k)}$. Or the solutions may be chosen as indicated in §29.

The system (40) has $w+2$ equations, and, when $w<n$, $\frac{1}{2}w(w-1)+w+1$ variables, and therefore $\frac{1}{2}(w+1)(w-2)$ solutions. When $w\leqq n$, the treatment is exactly the same as for the general case (§30), but it is not thought to be necessary to write out the equations corresponding to (32). They are obtained from (40) by neglecting those terms in which $\frac{\partial f}{\partial z^{(k)}}$ occurs, where $k\leqq n$. The first two equations of (40) show that every absolute invariant must be homogeneous of degree zero in $z, z', \ldots, z^{(w)}$, and isobaric of weight zero, as in the general case.

The Relative Invariants.

§37. Relative invariants, that is, invariant equations, may be formed either by equating an absolute invariant to a constant, or by forming determinants of the matrix of the coefficients of (39) or (40), as indicated by the general theory. This latter process is long, and the writer has found no method of applying it to the general case.

§38. We may use another method to find the relative invariants which are reproduced save as to a factor of known form, and §16 shows us that the form of the factor is *always* known. Let us seek, as Forsyth does, those relative invariants, V, for which

$$V_1=\left(\frac{d\chi}{dx}\right)^{\nu}V, \tag{41}$$

where ν is an integer. Such invariants V cannot be the *absolute* invariants of any but a trivial subgroup of (34). For suppose that V were an absolute invariant. Then

$$\frac{d\chi}{dx}=1^{\frac{1}{\nu}}$$

and

$$x_1=\chi(x)=1^{\frac{1}{\nu}}\,x+\text{const.}$$

As we are dealing with groups containing the identical transformation, this becomes

$$x_1=x+\text{const.},$$
$$\delta x_1=\delta t,$$
$$\xi=1,\quad \xi'=0,\quad \phi=C,\quad \delta y=Cy\,\delta t\ (\text{by }(34)).$$

(37) then gives $\delta p_j^{(k)} = 0$. $p_j^{(k)}$ is therefore not transformed, and *every* function of $p_j^{(k)}$ is an invariant for this transformation

$$\begin{cases} x_1 = x + \alpha, \\ y_1 = \beta y. \end{cases}$$

The relative invariants in question cannot, therefore, be found as the absolute invariants of *any subgroup* of (34).

§39. They may be found, however, as the solutions of certain partial differential equations. If in (41) we write

$$\chi = x + \delta t . \xi(x),$$

so as to get the infinitesimal transformation, we find

$$\begin{aligned} &V_1 = (1 + \xi' \delta t)^{\nu} V, \\ &\delta V = \nu \xi' V . \delta t = Z^{(w_1 w_2)} V . \delta t \text{ by (38)}, \\ \therefore\ &Z^{(w_1 w_2)} V = \nu V . \xi'. \end{aligned}$$

This equation must subsist identically for every ξ. It therefore breaks up into a number of equations, which are the same as (40) with the exception of the equation obtained by equating the coefficient of ξ' to zero. This equation would be replaced by

$$\nu f = \sum_{1}^{w}{}_{k} k z^{(k)} \frac{\partial f}{\partial z^{(k)}} + \sum_{2}^{\left\{ w \atop n \right.}{}_{j} \sum_{0}^{w-j}{}_{k} (k+j) p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}} \equiv Wf.$$

This equation tells us that f must be an isobaric function of weight ν (cf. Theorem I). If f be isobaric of weight ν, then $f^{\frac{\mu}{\nu}}$ is isobaric of weight μ, and is a solution of the other equations if f is. The ν of the foregoing equation may therefore be replaced by any other number, zero excepted, for instance by $\nu = 1$. The solutions seem to be simplest when we set $\nu = w$. We have then

Equations to be satisfied by the relative invariants for which $V_1 = \chi'^{w} V$, for the subgroup, $\phi' = \frac{n-1}{2} \xi''$.

$$\left. \begin{aligned} &wf = \sum_{1}^{w}{}_{k} k z^{(k)} \frac{\partial f}{\partial z^{(k)}} + \sum_{2}^{\left\{ w \atop n \right.}{}_{j} \sum_{0}^{w-j}{}_{k} (k+j) p_j^{(k)} \frac{\partial f}{\partial p_j^{(k)}}, \\ &0 = \sum_{0}^{w}{}_{k} z^{(k)} \frac{\partial f}{\partial z^{(k)}}, \\ &0 = 2X'_{s+1} f, \qquad (s = 1, 2, \ldots, w) \end{aligned} \right\} \qquad (42)$$

where $X'_{s+1} f$ is given by (40).

§40. If these equations be rendered homogeneous by substituting a relation of the form

$$F(f, \dots . z^{(l)}, \dots . p_j^{(k)} \dots .) = 0,$$

it is readily shown that the resulting system of equations form a complete system with $w + 2$ equations and one more variable than (40). It has therefore one more solution than (40). All but one of these solutions may be chosen free from f, being the solutions of (40). One solution F must contain f, and therefore there is at least one solution of the system (42). If this solution be multiplied by any solution of (40) we evidently get a new solution of (42). Conversely, the quotient of any two solutions of (42) is a solution of (40). These two facts together show that the number of solutions of (42) is exactly one greater than the number of solutions of (40), and *all* the solutions of (40) may be expressed as quotients of solutions of (42). (Cf. Theorem III.)

§41. The number of solutions of (42) which are free from $z^{(l)}$ is one greater than the number of solutions of (39), therefore at least $\frac{1}{2} w (w - 3)$. When $w = 3$ the equations obtained from (42) are not independent, there is one relation between them. They are

$$\begin{aligned}
3f &= 2p_2 \frac{\partial f}{\partial p_2} + 3p_2' \frac{\partial f}{\partial p_2'} + 3p_3 \frac{\partial f}{\partial p_3}, \\
0 &= -4p_2 \frac{\partial f}{\partial p_2'} - 6p_2 \frac{\partial f}{\partial p_3}, \\
0 &= -\frac{n+1}{3} \frac{\partial f}{\partial p_2}, \\
0 &= -\frac{n+1}{3} \frac{\partial f}{\partial p_2'} - \frac{n+1}{2} \frac{\partial f}{\partial p_3}.
\end{aligned}$$

The one solution is $f = p_3 - \frac{3}{2} p_2'$. The other solutions of (42) for $w = 3$ are

$$\frac{1}{z^3} \left[(n-1) z'' z - (n-2) z'^2 + \frac{3(n-1)^2}{n+1} p_2 z^2 \right]^{\frac{3}{2}},$$

$$\frac{1}{z^3} \left[(n-1)^2 z''' z - 3(n-1)(n-3) z'' z' z + 2(n-2)(n-3) z'^3 \right.$$
$$\left. + \frac{12(n-1)^2}{n+1} p_2 z' z^2 + \frac{2(n-1)^3}{n+1} p_3 z^3 \right].$$

Either of the last two solutions divided by $p_3 - \frac{3}{2} p_1'$ is a solution of (40), and of course may be obtained from the solutions of (30) found in §32 by setting $a_1 = 0$.

CHAPTER 5.

Differential Equation in Canonical Form.

§42. Laguerre has shown* that the general equation (1) may, by suitably choosing the functions χ and Ψ of the transformation (11), always be transformed into an equation in which $a_1 = a_2 = 0$. Let us take this, with Forsyth, as the canonical form of (1). We then have to find the invariants of

$$u^{(n)} + \sum_3^n s \frac{n!}{(n-s)!\; s!} q_s u^{(n-s)} = 0. \tag{43}$$

The transformations which leave (43) invariant must be among those defined by (34), since $q_1 = 0$. Therefore we may use (37) to compute δp_2. It gives

$$\delta p_2 = -\delta t \left[2p_2\xi' + \frac{n+1}{6}\xi''' \right].$$

This must vanish when $p_2 = 0$. Hence the transformations which leave (43) invariant are defined by

$$\xi''' = 0, \quad \phi' = \frac{n-1}{2}\xi'', \tag{44}$$

or $\qquad \delta x = (\alpha_0 + 2\alpha_1 x + \alpha_2 x^2)\,\delta t, \quad \delta u = [C + (n-1)(\alpha_1 + \alpha_2 x)]\, u\delta t.$

Our group therefore has the following infinitesimal transformations:

$$\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, \quad u\frac{\partial f}{\partial u}, \quad x^2\frac{\partial f}{\partial x} + (n-1)\,xu\frac{\partial f}{\partial u}. \tag{45}$$

The infinite group (11) for this case reduces to a finite group with 4 parameters. The finite equations of this group are

$$x_1 = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad u_1 = \frac{u}{(\gamma x + \delta)^{n-1}}. \tag{46}$$

Equation (35) now reduces to

$$\delta u^{(k)} = \left[\left(C + \frac{n-2k-1}{2}\xi' \right) u^{(k)} + \frac{k(n-k)}{2} u^{(k-1)} \right] \delta t, \tag{47}$$

and (37) becomes

$$\delta q_j^{(k)} = \{ -(k+j)\, q_j^{(k)}\zeta' - \tfrac{1}{2}\,[k(k+2j-1)\, q_j^{(k-1)} + j(j-1)\, q_{j-1}^{(k)}]\, \xi'' \}\, \delta t. \tag{48}$$

$$(j = 3, 4, \ldots, n; \; k = 0, 1, 2, \ldots; \; q_2 = 0)$$

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* Comptes Rendus, vol. 88 (1879), p. 226; see also Forsyth, Phil. Trans., 1888, I, p. 403.

§43. We find then

Equations to be satisfied by invariants of (43) *involving only* $q_j^{(k)}$, *where* $k+j \leqq w$.

$$\left.\begin{aligned}
0 = A_1^{(w)}f &\equiv \sum_{3}^{\left\{\substack{w\\ n}\right.}{}^{j} \sum_{0}^{w-j}{}^{k} (k+j)\, q_j^{(k)} \frac{\partial f}{\partial q_j^{(k)}}\,. \\
0 = A_2^{(w)}f &\equiv \sum_{3}^{\left\{\substack{w-1\\ n}\right.}{}^{j} \sum_{1}^{w-1}{}^{k} k(k+2j-1)\, q_j^{(k-1)} \frac{\partial f}{\partial q_j^{(k)}} + \sum_{4}^{\left\{\substack{w\\ n}\right.}{}^{j} \sum_{0}^{w-j}{}^{k} j(j-1)\, q_{j-1}^{(k)} \frac{\partial f}{\partial q_j^{(k)}}.
\end{aligned}\right\} \quad (49)$$

Equations to be satisfied by invariants of (43) *involving* $q_j^{(k)}$, $(k+j \leqq w)$, *and* $u^{(l)}\,(l \leqq w)$.

$$\left.\begin{aligned}
0 = X_0^{(w)}f &\equiv \sum_{0}^{\left\{\substack{n-1\\ w}\right.}{}^{k} u^{(k)} \frac{\partial f}{\partial u^{(k)}}\,, \\
0 = X_1^{(w)}f &\equiv \sum_{1}^{\left\{\substack{n-1\\ w}\right.}{}^{k} k u^{(k)} \frac{\partial f}{\partial u^{(k)}} + \sum_{3}^{\left\{\substack{w\\ n}\right.}{}^{j} \sum_{0}^{w-j}{}^{k} (k+j)\, q_j^{(k)} \frac{\partial f}{\partial q_j^{(k)}}\,, \\
0 = X_2^{(w)}f &\equiv \sum_{1}^{\left\{\substack{n-1\\ w}\right.}{}^{k} k(n-k)\, u^{(k-1)} \frac{\partial f}{\partial u^{(k)}} - A_2^{(w)}f,
\end{aligned}\right\} \quad (50)$$

where $A_2^{(w)}f$ is given by (49).

§44. The number of solutions of (49) when $w \leqq n$ is $\frac{1}{2}(w-2)(w-1)-2 = \frac{1}{2}(w^2-3w-2)$, for the equations are independent. When $w > n$, the number of solutions is $(n-2)w - \frac{1}{2}(n^2-n+2)$. The number of solutions of (50) when $w < n$ is $\frac{1}{2}(w-2)(w+1)$, and when $w \geqq n$ is $(n-2)w - \frac{1}{2}(n^2-3n+4)$.

The number of solutions of (50) which must contain $u^{(l)}$ is w or $n-1$, according as $w < n$ or $w \geqq n$. All except *two* of these solutions may be chosen free from $q^{(k)}$. They are therefore the same for *all* equations (43) of the n^{th} order. Hence *a linear differential equation of the form* (43) *has two, and only two, characteristic covariants.*

By writing out the first few terms of (50) it is easily proved that these two covariants may be chosen as

$$\frac{6q_3u' + (n-1)\,q_3'u}{uq_3^{\frac{4}{3}}} \quad \text{and} \quad \frac{(n-1)\,u''u - (n-2)\,u'^2}{u^2q_3^{\frac{2}{3}}}\,. \qquad (51)$$

Complete Solution of the System (50).

§45. This problem may be divided into two parts, viz.

1. The problem of finding all the solutions of (49).
2. The problem of finding all those solutions of (50) which contain $u^{(l)}$ only.

These two classes of solutions, together with the two solutions (51), constitute the complete solution of (50).

§46. Let us first find all those solutions of (50) which contain $u^{(l)}$ only. These are called by Forsyth* the "identical covariants." They are the solutions of

$$\sum_{0}^{\left\{\substack{n-1\\w}\right.} {}^{k}\, u^{(k)} \frac{\partial f}{\partial u^{(k)}} = 0, \quad \sum_{1}^{\left\{\substack{n-1\\w}\right.} {}^{k}\, k u^{(k)} \frac{\partial f}{\partial u^{(k)}} = 0, \quad \sum_{1}^{\left\{\substack{n-1\\w}\right.} {}^{k}\, k(n-k) u^{(k-1)} \frac{\partial f}{\partial u^{(k)}} = 0. \quad (52)$$

We can easily find all the solutions of these equations by making use of the general principles given in §§18, 19. The group (46) is evidently a subgroup of (11) for which

$$\Psi = C\chi'^{\frac{n-1}{2}},$$

and therefore if we can find one relative invariant the rest may be found by using (22). The matrix of the three equations (52) is

$$\left\| \begin{matrix} u & u' & u'' & u''' & \cdots\cdot \\ 0 & u' & 2u'' & 3u''' & \cdots\cdot \\ 0 & (n-1)u & 2(n-2)u' & 3(n-3)u'' & \cdots\cdot \end{matrix} \right\|.$$

Equating the first three-rowed determinant to zero gives

$$U_2 = (n-1)u''u - (n-2)u'^2$$

as a relative invariant, and this is the *only* relative invariant which can be found by forming determinants. Now using (22) gives

$$\frac{\frac{n-1}{2} u U_2' - \left(2\frac{n-1}{2} - 2\right) u' U_2}{U_2^{\frac{3}{2}}},$$

or
$$\Phi_3 = \frac{(n-1)^2 u''' u^2 - 3(n-1)(n-3) u'' u' u + 2(n-2)(n-3) u'^3}{U_2^{\frac{3}{2}}}$$

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* Phil. Trans., 1888, I, p. 428.

as an absolute invariant. Then by (21)

$$\Phi_4 = \left(\frac{u^2}{U_2}\right)^{\frac{1}{2}} \frac{d\Phi_3}{dx} = \Delta\Phi_3$$

is a second absolute invariant, and continuing the process we find

$$\Phi_3, \ \Delta\Phi_3, \ \Delta^2\Phi_3, \ldots, \Delta^{w-3}\Phi_3,$$

as all the solutions of (50) involving only u and its derivatives, the last one only involving $u^{(w)}$. These solutions are evidently independent. This method does not seem to readily lend itself to the computation of all the solutions in explicit form. We therefore proceed in the following manner:

§47. Let us assume at first that $w \leqq n - 1$. The first of the equations (52) tells us that f is homogeneous of degree zero in $u^{(l)}$, and the second that f is isobaric of weight zero in $u^{(l)}$. u is a solution of the second and third equations, and $U_2 = (n-1)u''u - (n-2)u'^2$ is a solution of the third. If then we can find an isobaric and homogeneous solution of the third equation, we can make it isobaric of weight zero by dividing by a suitable power of U_2, which is homogeneous, and isobaric of weight 2. We can then make the function homogeneous of degree zero by multiplying by a power of u. The result will be a solution of (52). (This process amounts to finding another relative invariant, and applying (21)). We have then to find a homogeneous and isobaric solution of the third equation of (52). There are evidently no solutions *linear* in $u^{(l)}$. There are, however, isobaric solutions which are homogeneous of the second degree in $u^{(l)}$. When $2j \leqq w$, and we write

$$F_{2j} = \sum_{0}^{j-1}{}^{i} \beta_i u^{(2j-i)} u^{(i)} + \tfrac{1}{2}\beta_j \left[u^{(j)}\right]^2,$$

where β_i is an undetermined constant, we have

$$\begin{aligned} X_2^{(w)} F_{2j} &= \sum_{1}^{j}{}^{k} k(n-k)\beta_k u^{(k-1)} u^{(2j-k)} + \sum_{j+1}^{2j}{}^{k} k(n-k)\beta_{2j-k} u^{(k-1)} n^{(2j-k)} \\ &= \sum_{1}^{j}{}^{k} \left[k(n-k)\beta_k + (2j-k+1)(n-2j+k-1)\beta_{k-1}\right] u^{(k-1)} u^{(2j-k)}. \end{aligned}$$

If now

$$\beta_k = -\frac{(2j-k+1)(n-2j+k-1)}{k(n-k)}\beta_{k-1},$$

or

$$\beta_k = (-1)^k \frac{(2j)!\ (n-2j+k-1)!\ (n-k-1)!}{(2j-k)!\ k!\ \ (n-2j-1)!\ (n-1)!}, \tag{53}$$

$$(k = 0,\ 1,\ 2,\ \ldots\ldots,\ j)$$

then we have

$$U_{2j} = \sum_{0}^{j-1}{}_{k}\, \beta_k u^{(2j-k)} u^{(k)} + \tfrac{1}{2}\beta_j \left[u^{(j)}\right]^2, \tag{54}$$

$$\left(j = 1,\ 2,\ \ldots\ldots \frac{w}{2} \text{ or } \frac{w-1}{2}\right)$$

is a solution of $X_2^{(w)}f = 0$. U_{2j} is isobaric of weight $2j$, and homogeneous of the second degree.

$$\Phi_{2j} = \frac{u^{2j-2}U_{2j}}{U_2^j}, \tag{55}$$

$$\left(j = 2,\ 3,\ \ldots\ldots \frac{w}{2} \text{ or } \frac{w-1}{2}\right)$$

is therefore a solution of (52). Since $2j \leqq w$, the number of such solutions is $\frac{w}{2} - 1$ or $\frac{w-3}{2}$, according as w is even or odd. There are in all $w - 2$ solutions involving $u^{(l)}$ alone. Hence we have still to find $\frac{w}{2} - 1$ or $\frac{w-1}{2}$ solutions of (52).

§48. When we attempt to apply the foregoing method to find solutions of $X_2^{(w)}f = 0$ which are isobaric of *odd* weight and homogeneous of the second degree, we find that all the coefficients are zero, and there are no such solutions. The next step would be to assume as solution an isobaric function homogeneous of the third degree, but the work of substituting such a function in the equation $X_2^{(w)}f = 0$ would be long. We might make use of the general method and choose solutions of the form

$$\frac{d\Phi_{2j}}{dx}\Big/\frac{d\Phi_3}{dx}, \qquad \left(j = 2,\ 3,\ \ldots\ldots \frac{w}{2} - 1,\ \text{or } \frac{w-1}{2}\right)$$

where Φ_3 has the value given in §46. This would give us a number of solutions which are evidently independent of those already found, and exactly equal in number to those we still need. They would therefore complete the solution. If, however, we take the solutions of this form, they are unnecessarily complicated. The numerator of the fraction would be u^{2j-3} multiplied into a function homogeneous of the *fifth* degree and isobaric of weight $2j + 1$. This function is a solution of $X_2^{(w)}f = 0$, but it is not the simplest solution which is of weight $2j+1$. We shall now prove that there are such solutions homogeneous of the *third* degree.

Theorem III applied to (55) shows that U_{2j} is a relative invariant. Then (22) shows that an absolute invariant is

$$\frac{(n-1)\,u\,U'_{2j} - 2\,(n-2j-1)\,u'\,U_{2j}}{u^{\frac{j-1}{j}}\;U_{2j}^{\frac{2j+1}{2j}}}\,.$$

When the value of U_{2j} from (54) is substituted here we get a solution of (50). If we write

$$U_{2j+1} = (n-1)\,u\,U'_{2j} - 2\,(n-2j-1)\,u'\,U_{2j}, \tag{56}$$

then

$$\Phi_{2j+1} = \frac{u^{2j-2}\,U_{2j+1}}{U_2^{\frac{2j+1}{2}}}, \qquad \left(j = 1,\ 2, \ldots\ldots,\ \frac{w}{2}-1 \text{ or } \frac{w-1}{2}\right) \tag{57}$$

is the solution of (50) which we sought. U_{2j+1} is isobaric of weight $2j+1$ and homogeneous of the third degree. (57) gives $\frac{w}{2}-1$ or $\frac{w-1}{2}$ solutions according as w is even or odd, and these, together with the solutions Φ_{2j}, $\frac{w}{2}-1$ or $\frac{w-3}{2}$ in number, are the $w-2$ independent solutions which involve $u^{(l)}$ alone.

The treatment of the case $w \leqq n$ is exactly the same as in the general case, and will not be further considered.

§49. Having now found all the solutions of (50) which necessarily involve u or its derivatives, we must next find all the solutions of (49). The equation $A_1^{(w)}f = 0$ simply shows that the solution must be isobaric of weight zero in the $q_j^{(k)}$'s. q_3 is a solution of the second equation. Therefore if we can find any other isobaric sulution of the second equation, we can make it isobaric of weight zero by dividing it by a suitable power of q_3 and thus get a solution of the system. Let us first assume $w \leqq n$. Then we may write $A_2^{(w)}f = 0$ in the form

$$0 = A_2^{(w)}f \equiv \sum_4^w{}_i \sum_3^{i-1}{}_j \left[(i-j)(i+j-1)\frac{\partial f}{\partial q_j^{(i-j)}} + j\,(j+1)\frac{\partial f}{\partial q_{j+1}^{(i-j-1)}}\right] q_j^{(i-j-1)}.$$

Let us see whether any solutions of the first degree in $q_j^{(k)}$ and isobaric of weight m exist. Assume

$$\Theta_m = \sum_0^{m-3}{}_s\, \alpha_{m,\,s}\, q_{m-s}^{(s)}. \qquad (4 \leqq m \leqq w)$$

Then

$$A_2^{(w)}\Theta_m = \sum_1^{m-1}{}_j \left[(m-j)(m+j-1)\,\alpha_{m,\,m-j} + j\,(j+1)\,\alpha_{m,\,m-j-1}\right] q_j^{(m-j-1)}.$$

Θ_m is therefore a solution of $A_2^{(w)}f = 0$ if

$$\alpha_{m,\,m-j} = -\frac{j\,(j+1)}{(m-j)(m+j-1)}\,\alpha_{m,\,m-j-1}, \qquad (j = 3,\ 4,\ \ldots\ldots,\ m-1)$$

$$\alpha_{m,\,s} = -\frac{(m-s)(m-s+1)}{s\,(2m-s-1)}\,\alpha_{m,\,s-1}, \qquad (s = 1,\ 2,\ \ldots\ldots,\ m-3)$$

$$= (-1)^s \frac{(m-2)!\ m!\ (2m-s-2)!}{(m-s-1)!\ (m-s)!\ (2m-3)!\ s!}\cdot\frac{\alpha_{m,\,0}}{2}.$$

The last equation is an identity for $s = 0$. If we choose $\alpha_{m,\,0} = 2$, then

$$\alpha_{m,\,s} = (-1)^s \frac{m!\ (m-2)!\ (2m-s-2)!}{(m-s)!\ s!\ (m-s-1)!\ (2m-3)!}, \qquad (58)$$

$$(s = 0,\ 1,\ 2,\ \ldots\ldots,\ m-3)$$

Therefore with this value of $\alpha_{m,\,s}$

$$\Theta_m = \sum_0^{m-3}{}_s\, \alpha_{m,\,s}\, q_{m-s}^{(s)}, \qquad (m = 4,\ 5,\ \ldots\ldots,\ w) \qquad (59)$$

is a solution of $A_2^{(w)}f = 0$. Then

$$J = \frac{\Theta_m}{q_3^{\frac{m}{3}}}, \qquad (m = 4,\ 5,\ \ldots\ldots,\ w) \qquad (60)$$

is a solution of (49). We have $w - 3$ such solutions. Direct integration of (49) shows that

$$J_3 = \frac{6q_3''\,q_3 - 7q_3'}{q_3^{\frac{8}{3}}} \qquad (61)$$

is a solution. From these $w - 2$ solutions all the others,

$$\tfrac{1}{2}(w^2 - 3w - 2) - (w - 2) = \tfrac{1}{2}(w^2 - 5w + 2)$$

in number, can be found by differentiation processes. We have simply to apply (21). If we define the operator ϑ by

$$\vartheta = \frac{1}{q_3^{\frac{1}{3}}}\,\frac{d}{dx}, \qquad (62)$$

then (21) gives us as absolute invariants (and hence solutions of (49)) the following set:

$$\left.\begin{array}{l} J_m,\ \vartheta J_m,\ \vartheta^2 J_m, \ldots\ldots, \vartheta^{w-m} J_m; \quad (m = 4, 5, \ldots\ldots, w) \\ J_3\ ,\ \vartheta J_3\ ,\ \vartheta^2 J_3\ , \ldots\ldots, \vartheta^{w-5} J_3 . \end{array}\right\} \tag{63}$$

These solutions are independent of one another, and their number is

$$(w-4) + [(w-3) + (w-4) + (w-5) + \ldots\ldots + 2 + 1] = \tfrac{1}{2}(w^2 - 3w - 2);$$

that is, (63) gives all the independent solutions of (49) for $w \leqq n$. If $w > n$, further differentiation of $I_3, \ldots\ldots, I_n$ is seen to give all the solutions.

§50. *To summarize, the complete solution of* (50) *consists of the following invariants:*

$$\frac{6q_3u' + (n-1)\,q_3'u}{q_3^{\frac{4}{3}}u} \text{ and } \frac{(n-1)\,u''u - (n-2)\,u'^2}{q_3^{\frac{2}{3}}u^2}, \tag{51}$$

$$\Phi_{2j} = \frac{u^{2j-2}U_{2j}}{U_2^j}, \qquad \left(j = 2, 3, \ldots\ldots, \frac{w}{2} \text{ or } \frac{w-1}{2}\right) \tag{55}$$

$$\Phi_{2j+1} = \frac{u^{2j-2}U_{2j+1}}{U_2^{\frac{2j+1}{2}}}, \quad \left(j = 1, 2, \ldots\ldots, \frac{w}{2} - 1 \text{ or } \frac{w-1}{2}\right) \tag{57}$$

$$J_3 = \frac{6q_3''q_3 - 7q_3'^2}{q_3^{\frac{8}{3}}}, \tag{61}$$

$$J_m = \frac{\Theta_m}{q_2^{\frac{m}{3}}}, \qquad (m = 4, 5, \ldots\ldots, w) \tag{60}$$

$$\left.\begin{array}{l} \vartheta J_m,\ \vartheta^2 J_m, \ldots\ldots, \vartheta^{w-m} J_m, \\ \vartheta J_3,\ \vartheta J_3\ , \ldots\ldots, \vartheta^{w-5} J_3\ . \end{array}\right\} (m = 4, 5, \ldots\ldots, w) \tag{63}$$

In these formulæ the quantities U_{2j}, U_{2j+1}, Θ_m and ϑ are defined by (54), (56), (59) and (62) respectively. These formulæ give the $\frac{1}{2}(w^2 - 3w - 2)$ invariants and the w covariants (which involve no letter of weight greater than $w \leqq n$) of the equation in canonical form

$$u^{(n)} + \sum_3^n s \frac{n\,!}{(n-s)\,!\ s\,!}\, q_s\, u^{(n-s)} = 0 \tag{43}$$

for all transformations which leave this equation invariant, namely, for the 4-parameter group

$$x_1 = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad u_1 = \frac{u}{(\gamma x + \delta)^{n-1}}. \tag{46}$$

Of these invariants, U_{2j}, U_{2j+1} and J_m are exactly the same as those given by Forsyth. The latter, however, gives instead of the derived set (63) another set of invariants, which of course must be functionally equivalent to (60), (61) and (63).

The Differential Parameter.

§51. In conclusion we shall show that all the different methods of which we have made use in deriving new invariants from known invariants by differentiation, are only special cases of what Lie calls the "Differential Parameter."*

Consider first the case of Theorem IV, where two invariants, I_1 and I_2 are known, and $\dfrac{dI_2}{dx}\bigg/\dfrac{dI_1}{dx}$ is shown to be invariant. The differential parameter is a function

$$\Omega\,(x,\, y,\, y',\, \ldots\,.\,,\, a^{(k)}\, \ldots\,.\, I_1,\, I_2,\, I_1',\, I_2'\, \ldots\,.)$$

such that when I_1 and I_2 are invariants, Ω also is invariant. We have

$$\delta\Omega = \frac{\partial\Omega}{\partial x}\,\delta x + \sum_0^w{}_k \frac{\partial\Omega}{\partial y^{(k)}}\,\delta y^{(k)} + \sum_1^{\left\{\substack{w\\n}\right.}{}_j \sum_0^{w-j}{}_k \frac{\partial\Omega}{\partial a_j^{(k)}}\,\delta a_j^{(k)} + \frac{\partial\Omega}{\partial I_1}\,\delta I_1 + \frac{\partial\Omega}{\partial I_2}\,\delta I_2$$
$$+ \frac{\partial\Omega}{\partial I_1'}\,\delta I_1' + \frac{\partial\Omega}{\partial I_2'}\,\delta I_2' + \ldots\,.$$
$$= X^{(w\,w)}\Omega\,.\,\delta t + \frac{\partial\Omega}{\partial I_1}\,\delta I_1 + \frac{\partial\Omega}{\partial I_2}\,\delta I_2 + \frac{\partial\Omega}{\partial I_1'}\,\delta I_1' + \frac{\partial\Omega}{\partial I_2'}\,\delta I_2' + \ldots\,.\,,$$

if we make use of the notation of (28). Moreover, we have

$$\delta I_1' = \delta\,\frac{dI_1}{dx} = \frac{d}{dx}\,\delta I_1 - I_1'\,\frac{d\delta x}{dx} = -\,I_1'\,\xi'\,\delta t, \text{ since } \delta I_1 = 0\,.$$

Since Ω is to be invariant when I_1 and I_2 are, we have $\delta\Omega = 0$ when $\delta I_1 = 0$, $\delta I_2 = 0$, and this for all values of ϕ and ξ. Hence, on substituting the values of $\delta I_1'$ and $\delta I_2'$, using (30) and (31) and equating the coefficients of $\phi^{(s)}$ and $\xi^{(s)}$ to zero, we get

$$0 = Y_s\Omega, \qquad (s = 0,\, 1,\, \ldots\,.\,,\, w)$$
$$0 = X_{s+1}\Omega, \qquad (s = 1,\, 2,\, \ldots\,.\,,\, w)$$
$$0 = -\,X_1\Omega + I_1'\,\frac{\partial\Omega}{\partial I_1'} + I_2'\,\frac{\partial\Omega}{\partial I_2'}\,.$$

This is a complete system of equations giving Ω. The system has all the solutions of (30) and two others, which are the differential parameters. One is evidently

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* Lie, "Continuierliche Gruppen," herausgegeben von Scheffers, pp. 670, 680, 739.

$$\Omega_1 = \frac{I_2'}{I_1'} = \frac{dI_2}{dx} \Big/ \frac{dI_1}{dx},$$

which is the result given in the corollary to Theorem IV. The other solution is

$$\Omega_2 = (2a_3 - 3a_2' - 6a_2 a_1 + 4a_1^3 + 6a_1 a_1' + a_1'')^{-\frac{1}{3}} I_1'.$$

No use has been made of this second solution in the earlier part of the foregoing paper, but in Chapter 5 it is used in formula (62).

Next take the case of §18, where two relative invariants, U and V, are known, each of degree λ and weight w. The ratio $I = U/V$ is then a known invariant. As before, let

$$\Omega(x, \ldots, y^{(l)}, \ldots, a_j^{(k)}, \ldots, U, V, I, I')$$

be the differential parameter. Then

$$\delta\Omega = X^{(ww)}\Omega \,.\, \delta t + \frac{\partial\Omega}{\partial U}\delta U + \frac{\partial\Omega}{\partial V}\delta V + \frac{\partial\Omega}{\partial I}\delta I + \frac{\partial\Omega}{\partial I_1'}\delta I'.$$

Considerations similar to those of §39 show that U and V are solutions of

$$\begin{aligned} 0 &= Y_s f, && (s = 1, 2, \ldots, w) \\ 0 &= X_{s+1} f, && (s = 1, 2, \ldots, w) \\ \lambda f &= Y_0 f, && \\ wf &= -X_1 f, && \end{aligned}$$

$$\begin{aligned} \therefore\ \delta U &= X^{(ww)} U \,.\, \delta t = (\phi Y_0 U + \xi' X_1 U)\,\delta t = (\lambda\phi - w\xi')\, U \,.\, \delta t, \\ \delta V &= (\lambda\phi - w\xi')\, V \,.\, \delta t. \end{aligned}$$

As before $\delta I' = -I'\xi'\,\delta t$. Then $\delta\Omega = 0$, $\delta I = 0$ gives

$$0 = \xi\frac{\partial\Omega}{\partial x} + \sum_0^w{}_s\, \phi^{(s)} Y_s\Omega + \sum_1^{w+1}{}_s\, \xi^{(s)} X_s\Omega + (\lambda\phi - w\xi')\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right) - \xi' I'\frac{\partial\Omega}{\partial I'},$$

This breaks up into

$$\begin{aligned} 0 &= Y_s\Omega, && (s = 1, 2, \ldots, w) \\ 0 &= X_{s+1}\Omega, && (s = 1, 2, \ldots, w) \\ 0 &= Y_0\Omega + \lambda\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right), && \\ 0 &= X_1\Omega - w\left(U\frac{\partial\Omega}{\partial U} + V\frac{\partial\Omega}{\partial V}\right) - I'\frac{\partial\Omega}{\partial I'}\,. && \end{aligned}$$

These equations have three more solutions than (30). One of them is U/V, the second is Ω_2 given earlier, and the third is the desired differential parameter, viz.

$$\Omega_3 = \left[\frac{y^\lambda}{V}\right]^{\frac{1}{w}} I,$$

This is the same as (21). It should be noticed that no use has been made of the fact that $I = U/V$, so that in the last formula I may be any invariant whatever and V any relative invariant of degree λ and weight w.

Lastly, consider the case of §19, and suppose that we know a single relative invariant, U, of degree λ and weight w. For this case $\phi = \gamma + \nu\xi'$, where γ is an arbitrary constant. The transformation becomes

$$X^{(ww)}f = \xi\,\frac{\partial f}{\partial x} + \gamma Y_0 f + \overset{w+1}{\underset{1}{\sum\nolimits_s}} (\nu Y_{s-1}f + X_s f)\,\xi^{(s)}.$$

Let us see whether there is a differential parameter of the form

$$\Omega\,(x, \ldots\ldots, y^{(l)}, \ldots\ldots a_j^{(k)}, \ldots\ldots, U,\ U')\text{-}$$

We have, as before,

$$\begin{aligned}\delta U &= (\lambda\phi - w\xi')\,U\delta t = [\lambda\gamma + (\lambda\nu - w)\,\xi']\,U\delta t,\\ \delta U' &= \frac{d}{dx}\,\delta U - U'\,\frac{d\delta x}{dx}\\ &= (\lambda\nu - w)\,\xi'' U\delta t + [\lambda\gamma + (\lambda\nu - w - 1)\,\xi']\,U'.\delta t.\end{aligned}$$

Substituting the values of δU and $\delta U'$, and equating the coefficients of γ and $\xi^{(s)}$ to zero gives

$$\begin{aligned}0 &= Y_0\Omega + \lambda U\frac{\partial\Omega}{\partial U} + \lambda U'\frac{\partial\Omega}{\partial U'},\\ 0 &= X_1\Omega - wU\frac{\partial\Omega}{\partial U} - (w+1)\,U'\frac{\partial\Omega}{\partial U'},\\ 0 &= \nu Y_1\Omega + X_2\Omega + (\lambda\nu - w)\,U\frac{\partial\Omega}{\partial U'},\\ 0 &= \nu Y_{s-1}\Omega + X_s\Omega. && (s = 3,\ 4, \ldots\ldots, w+1)\end{aligned}$$

Expanding by using (30) gives

$$\begin{aligned}0 &= y\,\frac{\partial\Omega}{\partial y} + y'\,\frac{\partial\Omega}{\partial y'} + \lambda U\frac{\partial\Omega}{\partial U} + \lambda U'\frac{\partial\Omega}{\partial U'} + \ldots\ldots,\\ 0 &= y'\,\frac{\partial\Omega}{\partial y'} + wU\frac{\partial\Omega}{\partial U} + (w+1)\,U'\frac{\partial\Omega}{\partial U'} + \ldots\ldots,\\ 0 &= \nu y\,\frac{\partial\Omega}{\partial y'} + (\lambda\nu - w)\,U\frac{\partial\Omega}{\partial U'} + \ldots\ldots,\end{aligned}$$

. . . .

where the dots indicate terms which do not involve derivatives with regard to y, y', U or U'. Neglecting these terms we get a complete system with the one solution

$$\Omega_4 = \frac{(\lambda\nu - w)\,U'y - \nu\,Uy'}{y^{\frac{w-\lambda}{w}}\,U^{\frac{w+1}{w}}}\,.$$

This is the same as (22). In computing the first two differential parameters, Ω_1 and Ω_2, no use whatever was made of the finite transformations of the group. For the last two, however, use was made of the fact that the factor of (17) is known.